Exercises in Fundamental Physics

(Undergraduate L3 – Graduate M1 Level)

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Contents

Abstract

This document gathers a selection of original exercises in fundamental physics, designed with a transversal and progressive perspective, from the third year of undergraduate studies to the first year of a Master's degree. Each exercise is accompanied by a detailed solution (when available), and is embedded in a rigorous historical, theoretical, or practical context. Topics covered include special relativity, quantum mechanics, statistical physics, electrodynamics, and incursions into mathematical physics. A classification by level is proposed to guide the reader's progression.

Chapter 1

Introduction

This document is a compilation of exercises in Fundamental Physics that I designed with passion, in the spirit of an end-of-L3 / M1 course, and beyond. The aim is twofold: to provide rigorous, inspiring problems that highlight the formal and conceptual beauty of physics, and to offer a solid foundation for students wishing to deepen their understanding of major classical and modern theories. I hope to share my enthusiasm for physics that goes beyond what is typically covered in class, drawing on concepts that span multiple areas of physics.

Each exercise involves specific concepts (indicated in parentheses, such as **(SR)** for Special Relativity, **(QM)** for Quantum Mechanics, etc.) and is gradually supplemented with a detailed correction, accessible by clicking on the "(Correction)" link. Exercises are rated with stars (see 2.1), and you are free to start with the one that intrigues you the most.

As a first-year Master's student in Fundamental Physics at Sorbonne University (Pierre and Marie Curie campus), I want this collection to remain dynamic: solutions will be added regularly. Lastly, in the correction section, by clicking on the exercise titles (either in the heading or at the beginning of the solution), you can return to the corresponding exercise.

I hope that by reading and working through these exercises, you will find as much enjoyment as I had in writing them.

Chapter 2

Information

2.1 Notations

1. Vector quantities are written in bold, except for the operator ∇, which is never in bold. Four-vector quantities (in relativity) are written with a Greek letter, in superscript for contravariant components, and in subscript for covariant ones.
   Example: $\mathbf{v}$ for velocity, ∇p for the pressure gradient (which is a vector!), and x^μ for space-time position in contravariant form. Conversely, in Quantum Mechanics, vectors are denoted using kets, and operators in bold.
   Example: $|\psi\rangle$ for a state vector ψ and $\mathbf{H}$ for the Hamiltonian.

2. The notation d denotes the differential operator.

3. The notation ∂_u implicitly means $\frac{\partial}{\partial u}$ if u is a variable, and $\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = \frac{\partial}{\partial x_\mu}$ in relativity.

4. $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$ in Cartesian coordinates, is an operator that properly defines the gradient, divergence, and curl. Indeed, ∇f is the gradient of f, $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$, and $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$. The operator $\partial_\mu \partial^\mu = \Box$ is the d'Alembertian, invariant under Lorentz transformations.

5. The notation $\dot{x}$ denotes a time derivative: $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. In a relativity exercise, the preferred notation will be $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, where τ is the proper time, and $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

6. The notation f' denotes a derivative with respect to the variable x, i.e., $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

7. The notation $[A]$ indicates the physical unit of the quantity A.

8. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}$ denote the sets of real, complex, and natural numbers, respectively.

9. The metric used in special relativity is $g_{\mu\nu} = (-,+,+,+)$. We also recall that $a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu$.

10. Stars indicate the difficulty level of the exercises, ranging from 1: ★ to 5 stars: ★★★★★. The difficulty assessment is based on the length, technical and mathematical complexity, and the academic level (L3, M1, M2) needed to be comfortable with the concepts involved.

11. The symbol $\triangle$ indicates that the solution is still being written.

2.2 Fundamental Constants

Constant	Exact value	Units
Planck constant	$h = 6.62607015 \times 10^{-34}$	$\mathrm{J\,s}$
Dirac constant	$\hbar = \frac{h}{2\pi} = 1.054571817 \times 10^{-34}$	$\mathrm{J\,s}$
Speed of light	$c = 299792458$	$\mathrm{m\,s^{-1}}$
Elementary charge	$e = 1.602176634 \times 10^{-19}$	C
Electron mass	$m_e = 9.1093837015 \times 10^{-31}$	kg
Proton mass	$m_p = 1.67262192369 \times 10^{-27}$	kg
Neutron mass	$m_n = 1.675 \times 10^{-27}$	kg
Vacuum permittivity	$\varepsilon_0 = 8.854187817 \times 10^{-12}$	$\mathrm{F\,m^{-1}}$
Vacuum permeability	$\mu_0 = 4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2}}$
Gravitational constant	$G = 6.67430 \times 10^{-11}$	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Boltzmann constant	$k_B = 1.380649 \times 10^{-23}$	$\mathrm{J\,K^{-1}}$
Avogadro number	$\mathcal{N}_A = 6.02214076 \times 10^{23}$	$\mathrm{mol^{-1}}$
Ideal gas constant	$R = 8.314462618$	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Reference temperature (0°C)	$T_0 = 273.15$	K
Sun's mass	$M_\odot = 1.98892 \times 10^{30}$	kg

Table 2.1: Fundamental physical constants with their exact values.

2.3 Formulary

2.3.1 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss's law)} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Absence of magnetic monopoles)} \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère-Maxwell law)} \tag{2.4}$$

$$\nabla \times \mathbf{A} = \mathbf{B}, \quad -\partial_t \mathbf{A} - \nabla \varphi = \mathbf{E} \quad \text{(Relation between the vector potential and the EM field)} \tag{2.5}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{(Electric displacement)} \tag{2.6}$$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \text{(Polarization)} \tag{2.7}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \text{(Auxiliary magnetic field)} \tag{2.8}$$

$$\mathbf{M} = \chi_m \mathbf{H} \quad \text{(Magnetization)} \tag{2.9}$$

$$\varepsilon_r = 1 + \chi_e, \quad \mu_r = 1 + \chi_m \quad \text{(Relations to susceptibilities)} \tag{2.10}$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{n} \quad \text{(Wave speed in the medium)} \tag{2.11}$$

$$\Box \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \quad \text{(Wave equation in vacuum)} \tag{2.12}$$

$$\gamma = \sigma + i\omega\varepsilon \quad \text{(Complex conductivity)} \tag{2.13}$$

$$P = \frac{q^2 a^2}{6\pi \varepsilon_0 c^3} \quad \text{(Larmor power)} \tag{2.14}$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{(Poynting vector)} \tag{2.15}$$

2.3.2 Special Relativity

$$E = \gamma mc^2 = \sqrt{p^2c^2 + m^2c^4} \quad \text{(Relativistic energy)} \tag{2.16}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{(Lorentz factor)} \tag{2.17}$$

$$x' = \gamma(x - vt) \quad \text{(Lorentz transformation)} \tag{2.18}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{(Time transformation)} \tag{2.19}$$

$$\beta = \frac{v}{c} \tag{2.20}$$

$$\mathbf{p} = \gamma m\mathbf{v} \quad \text{(Relativistic momentum vector)} \tag{2.21}$$

$$\mathbf{p} = \hbar\mathbf{k} \quad \text{(Photon momentum vector)} \tag{2.22}$$

2.3.3 Quantum Mechanics

$$\mathbf{P} = -i\hbar\nabla \quad \text{(Momentum operator)} \tag{2.23}$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathbf{H}|\psi\rangle \quad \text{(Schrödinger equation)} \tag{2.24}$$

$$[\mathbf{X}_i, \mathbf{P}_j] = i\hbar\delta_{ij} \quad \text{(Canonical commutation relation)} \tag{2.25}$$

$$\langle\psi|\mathbf{A}|\psi\rangle = \langle A\rangle \quad \text{(Expectation value)} \tag{2.26}$$

$$(\Delta A)^2 = \langle\psi|(\mathbf{A} - \langle A\rangle)^2|\psi\rangle \quad \text{(Variance of an observable)} \tag{2.27}$$

$$\Delta A\,\Delta B \geq \frac{1}{2}|\langle\psi|[\mathbf{A},\mathbf{B}]|\psi\rangle| \quad \text{(Heisenberg uncertainty inequality)} \tag{2.28}$$

$$\mathbf{U}(t) = e^{-i\mathbf{H}t/\hbar} \quad \text{(Unitary time evolution)} \tag{2.29}$$

$$\mathbf{H}|E_n\rangle = E_n|E_n\rangle \quad \text{(Stationary eigenstates)} \tag{2.30}$$

$$\mathbb{P}(a_n) = |\langle a_n|\psi\rangle|^2 \quad \text{(Born rule probability)} \tag{2.31}$$

$$\mathbf{X} = \sqrt{\frac{\hbar}{2m\omega}}(\mathbf{a} + \mathbf{a}^\dagger), \quad \mathbf{P} = i\sqrt{\frac{\hbar m\omega}{2}}(\mathbf{a}^\dagger - \mathbf{a}) \quad \text{(Harmonic oscillator)} \tag{2.32}$$

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1} \quad \text{(Commutator)} \tag{2.33}$$

$$\mathbf{H} = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right) \quad \text{(Oscillator Hamiltonian)} \tag{2.34}$$

$$\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}, \quad \mathbf{N}|n\rangle = n|n\rangle \quad \text{(Number operator)} \tag{2.35}$$

$$\mathbf{L}_i = \varepsilon_{ijk}\mathbf{X}_j\mathbf{P}_k \quad \text{(Orbital angular momentum)} \tag{2.36}$$

$$[\mathbf{J}_i, \mathbf{J}_j] = i\hbar\varepsilon_{ijk}\mathbf{J}_k \quad \text{(Lie algebra of } SU(2)\text{)} \tag{2.37}$$

$$[\mathbf{H}, \mathbf{A}] = 0 \Rightarrow \mathbf{A} = \text{constant of motion} \quad \text{(Symmetry and conservation)} \tag{2.38}$$

2.3.4 Statistical Physics

Canonical ensemble (system in contact with a thermostat, with fixed number of particles)

$$\beta = \frac{1}{k_B T} \quad \text{(Temperature energy)} \tag{2.39}$$

$$Z = \sum_n e^{-\beta E_n} \quad \text{(Partition function)} \tag{2.40}$$

$$P_n = \frac{e^{-\beta E_n}}{Z} \quad \text{(Occupation probability of level } n\text{)} \tag{2.41}$$

$$\langle E \rangle = \sum_n E_n P_n = -\frac{\partial \ln Z}{\partial \beta} \quad \text{(Mean energy)} \tag{2.42}$$

$$\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} \quad \text{(Energy fluctuation)} \tag{2.43}$$

$$S = -k_B \sum_n P_n \ln P_n \quad \text{(Shannon statistical entropy)} \tag{2.44}$$

$$F = -k_B T \ln Z \quad \text{(Helmholtz free energy)} \tag{2.45}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \quad \text{(Link with thermodynamics)} \tag{2.46}$$

Grand canonical ensemble (system in contact with a particle and heat reservoir):

$$\mathcal{Z} = \sum_{N=0}^{\infty} \sum_n e^{-\beta(E_{n,N} - \mu N)} \quad \text{(Grand partition function)} \tag{2.47}$$

$$\mathcal{Z} = \prod_i \xi_i \quad \text{(Factorization over states)} \tag{2.48}$$

$$\xi_i = \sum_{n_i} e^{-\beta(\varepsilon_i - \mu) n_i} \quad \text{(Partition function for state } i\text{)} \tag{2.49}$$

$$\mathcal{J} = -k_B T \ln \mathcal{Z} \quad \text{(Grand potential)} \tag{2.50}$$

$$\mathcal{J} = -k_B T \sum_i \ln \xi_i \quad \text{(Grand potential, factorized form)} \tag{2.51}$$

$$P = -\left(\frac{\mathcal{J}}{V}\right) \quad \text{(Pressure)} \tag{2.52}$$

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial \ln \mathcal{Z}}{\partial \mu} \quad \text{(Mean number of particles)} \tag{2.53}$$

$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} + \mu \langle N \rangle \quad \text{(Mean energy)} \tag{2.54}$$

$$S = -\left(\frac{\partial \mathcal{J}}{\partial T}\right)_{V,\mu} \quad \text{(Entropy)} \tag{2.55}$$

$$F = \langle E \rangle - TS = \mathcal{J} + \mu \langle N \rangle \quad \text{(Link with free energy)} \tag{2.56}$$

2.3.5 Analytical Mechanics

$$\mathcal{L} = T - V \quad \text{(Lagrangian)} \tag{2.57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \quad \text{(Lagrange equations)} \tag{2.58}$$

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F(q,t)}{\mathrm{d}t} \quad \text{(Non-uniqueness of the Lagrangian)} \tag{2.59}$$

$$\mathcal{S}[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q,\dot{q},t)\,\mathrm{d}t \quad \text{(Action)} \tag{2.60}$$

$$\delta\mathcal{S} = 0 \quad \text{(Principle of least action)} \tag{2.61}$$

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \quad \text{(Conjugate momentum)} \tag{2.62}$$

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \Rightarrow p_\alpha = \text{const.} \quad \text{(Cyclic variable)} \tag{2.63}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \sum_\alpha \dot{q}_\alpha p_\alpha - \mathcal{L} = \text{const.} \quad \text{(Beltrami identity)} \tag{2.64}$$

$$\mathcal{H}(q,p,t) = \sum_\alpha p_\alpha \dot{q}_\alpha - \mathcal{L} \quad \text{(Hamiltonian)} \tag{2.65}$$

$$\dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \quad \text{(Hamilton's equations)} \tag{2.66}$$

$$\{f,g\} = \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right) \quad \text{(Poisson bracket)} \tag{2.67}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f,\mathcal{H}\} + \frac{\partial f}{\partial t} \quad \text{(Time evolution)} \tag{2.68}$$

$$Q_\alpha = \frac{\partial F}{\partial P_\alpha}, \quad p_\alpha = \frac{\partial F}{\partial q_\alpha} \quad \text{(Canonical transformation via } F_2(q,P,t)\text{)} \tag{2.69}$$

$$\mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) + \frac{\partial F}{\partial t} \quad \text{(New Hamiltonian)} \tag{2.70}$$

2.3.6 Subatomic Physics

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \quad \text{(Elementary solid angle)} \tag{2.71}$$

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \quad \text{(Total cross section)} \tag{2.72}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\varphi)|^2 \quad \text{(Differential cross section)} \tag{2.73}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\sin\theta}\left|\frac{\mathrm{d}b^2}{\mathrm{d}\theta}\right| \quad \text{(Classical differential cross section)} \tag{2.74}$$

$$B = \left[Zm_p + (A-Z)m_n - M(A,Z)\right]c^2 \quad \text{(Nuclear binding energy)} \tag{2.75}$$

$$Q = \left[m_{\text{initial}} - m_{\text{final}}\right]c^2 \quad \text{(Energy released in a reaction)} \tag{2.76}$$

$$N(t) = N_0 e^{-\lambda t} \quad \text{(Mean radioactive decay law)} \tag{2.77}$$

$$\tau = \frac{1}{\lambda}, \quad T_{1/2} = \frac{\ln 2}{\lambda} \quad \text{(Mean lifetime and half-life)} \tag{2.78}$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -\lambda_i N_i + \lambda_{i-1}N_{i-1} \quad \text{(Decay chain)} \tag{2.79}$$

2.3.7 Wave Optics

$$j^2 = -1 \quad \text{(Imaginary unit)} \tag{2.80}$$

$$\psi(x) = \psi_0 e^{jk\delta} \quad \text{(Monochromatic plane wave)} \tag{2.81}$$

$$\mathrm{d}\psi = \psi_0 e^{j\varphi(x)}\mathrm{d}x \quad \text{(Diffracted field element)} \tag{2.82}$$

$$\psi(M) = \int \psi_0(x)e^{j\varphi(x)}\mathrm{d}x \quad \text{(Diffracted field – Fresnel integral)} \tag{2.83}$$

$$\varphi(x) = \frac{(x-x')^2}{2z} \quad \text{(Phase in the Fresnel approximation)} \tag{2.84}$$

$$I(x) = I_0\left[1 + \cos\left(\frac{2\pi}{\lambda}\delta(x)\right)\right] \quad \text{(Two-wave interference)} \tag{2.85}$$

$$I = \left|\int_A^B \mathrm{d}\psi\right|^2 \quad \text{(Superposition principle – intensity)} \tag{2.86}$$

$$L_{AB} = \int_A^B n(\mathbf{r})\mathrm{d}s \quad \text{(Optical path)} \tag{2.87}$$

$$\varphi = k\delta = \frac{2\pi}{\lambda}L_{AB} \quad \text{(Associated phase shift)} \tag{2.88}$$

$$i = \frac{\lambda D}{a} \quad \text{(Fringe spacing in Fraunhofer approximation)} \tag{2.89}$$

2.3.8 Thermodynamics

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N \quad \text{(First law)} \quad (2.90)$$

$$\mathrm{d}S \geq \frac{\delta Q}{T} \quad \text{(Second law)} \quad (2.91)$$

$$F = U - TS \quad \text{(Helmholtz free energy)} \quad (2.92)$$

$$G = U + pV - TS = \mu N \quad \text{(Gibbs free energy)} \quad (2.93)$$

$$H = U + pV \quad \text{(Enthalpy)} \quad (2.94)$$

$$pV = Nk_BT = nRT \quad \text{(Ideal gas law)} \quad (2.95)$$

$$U = \frac{f}{2}Nk_BT \quad \text{(Internal energy, } f \text{ degrees of freedom)} \quad (2.96)$$

$$= \frac{3}{2}Nk_BT \quad \text{(Monoatomic ideal gas)} \quad (2.97)$$

$$= \frac{5}{2}Nk_BT \quad \text{(Diatomic ideal gas at high } T\text{)} \quad (2.98)$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{f}{2}Nk_B \quad \text{(Heat capacity at constant volume)} \quad (2.99)$$

$$C_P = C_V + Nk_B = \frac{f+2}{2}Nk_B \quad \text{(Heat capacity at constant pressure)} \quad (2.100)$$

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} \quad \text{(Adiabatic index)} \quad (2.101)$$

$$\mu = \left(\frac{\partial G}{\partial N}\right)_{T,p} \quad \text{(Chemical potential)} \quad (2.102)$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T \quad \text{(Pressure from free energy)} \quad (2.103)$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \quad \text{(Entropy from free energy)} \quad (2.104)$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \quad \text{(Maxwell relation)} \quad (2.105)$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V \quad \text{(Maxwell relation)} \quad (2.106)$$

$$\left(\frac{\partial U}{\partial S}\right)_V = T \quad \text{(Definition of temperature)} \quad (2.107)$$

$$\left(\frac{\partial U}{\partial V}\right)_S = -p \quad \text{(Definition of pressure)} \quad (2.108)$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g \quad \text{(Hydrostatic equilibrium)} \quad (2.109)$$

$$p(z) = p_0 e^{-\frac{mgz}{k_BT}} \quad \text{(Isothermal atmosphere)} \quad (2.110)$$

Cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{2.111}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{2.112}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_r \tag{2.113}$$

$$+ \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta \tag{2.114}$$

$$+ \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_z \tag{2.115}$$

Spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \tag{2.116}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{2.117}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{e}_r \tag{2.118}$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{e}_\theta \tag{2.119}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\phi \tag{2.120}$$

2.3.9 General Relativity

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad \text{(Spacetime interval)} \tag{2.121}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \qquad \text{(Christoffel symbol)} \tag{2.122}$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \qquad \text{(Riemann tensor)} \tag{2.123}$$

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \qquad \text{(Ricci tensor)} \tag{2.124}$$

$$R = g^{\mu\nu}R_{\mu\nu} \qquad \text{(Ricci scalar)} \tag{2.125}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \qquad \text{(Einstein tensor)} \tag{2.126}$$

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x + S_{\mathrm{mat}} \qquad \text{(Einstein-Hilbert action)} \tag{2.127}$$

$$\delta S = 0 \Rightarrow G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \text{(Einstein field equations)} \tag{2.128}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{mat}}}{\delta g^{\mu\nu}} \qquad \text{(Energy-momentum tensor)} \tag{2.129}$$

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \qquad \text{(Geodesic equation)} \tag{2.130}$$

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \text{(Local energy conservation)} \tag{2.131}$$

$$\sqrt{-g}\,\mathrm{d}^4x \qquad \text{(Invariant volume element)} \tag{2.132}$$

$$\det(g_{\mu\nu}) = g \qquad \text{(Determinant of the metric)} \tag{2.133}$$

(2.134)

2.3.10 Trigonometric identities

$$\sin^2(\theta) + \cos^2(\theta) = 1, \quad 1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)}. \tag{2.135}$$

Addition formulas

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b), \tag{2.136}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b). \tag{2.137}$$

Double-angle formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \tag{2.138}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta). \tag{2.139}$$

These formulas are very useful for variable changes in integration.

Expressions of $\sin(x)$, $\cos(x)$, and $\tan(x)$ in terms of $t = \tan\left(\frac{x}{2}\right)$

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}. \tag{2.140}$$

Variable substitution $t = \tan\left(\frac{x}{2}\right)$

This change of variable is often used to simplify trigonometric integrals. We also have:

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t. \tag{2.141}$$

2.4 Legend of thematic notations

- **(SR)**: Special Relativity
- **(QM)**: Quantum Mechanics
- **(EM)**: Electromagnetism
- **(AM)**: Analytical Mechanics
- **(SM)**: Statistical Mechanics
- **(SP)**: Subatomic Physics
- **(WO)**: Wave Optics
- **(TD)**: Thermodynamics
- **(GR)** : General Relativity

2.5 Suggested paths depending on your level

To help readers navigate this dense collection of exercises, here are a few suggested paths based on your level and goals. Of course, every student is free to explore the problems that inspire them.

Level	Recommended exercises
Early Bachelor Year 3	3.1 – Two-body problem 3.2 – Rutherford cross section 3.4 – Pulsed magnetic field machine 3.13 – Electrodynamic instability of the classical atom
End of Bachelor / Beginning of Master 1	3.3 – Cherenkov effect 3.5 – Metric on a sphere 3.6 – Blackbody radiation 3.10 – Hydrogen atom and radial equation 3.12 – Pöschl–Teller potential 3.14 - Geodesics in a Dispersive Optical Medium 3.15 - Bose-Einstein Condensation 3.16 - Decay Chain
Advanced Master 1	3.7 – Minimization of gravitational potential 3.8 – Relativistic charged particle 3.9 – Relativistic hydrodynamics 3.11 – Towards a relativistic formalism 3.17 - From the Principle of Least Action to Einstein's Equations

Chapter 3

Exercises

This collection of exercises was designed with the ambition to go beyond mere mechanical practice of methods. Each problem aims to highlight a certain form of mathematical elegance or physical depth — a careful eye will discover, behind the equations and techniques, a subtle coherence, sometimes even a formal beauty. Some exercises are demanding, both in their length and structure: they are sometimes inspired by competitive exams or realistic physical situations, and may require several hours of reflection. Their goal is not only to reinforce technical skills, but to make one feel, through progressive resolution, the deep unity between mathematical rigor and the physical reality it describes. This chapter is dynamic: new problems will be regularly added in the same spirit of elegance, clarity, and depth.

3.1 Two-body problem and quantization of the Bohr atom [3] (AM) ★★★

(Solution)

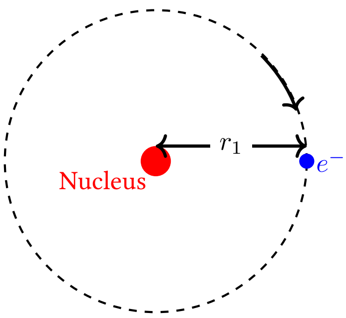

Figure 3.1: Diagram of the Bohr atom.

Consider a system of two particles with masses m_1 and m_2 interacting via a central potential $V(r) = -\frac{C}{r} = -\frac{\vartheta^2}{r}$[1], where r is the distance between the two particles and C is a real constant. Here we use the Coulomb potential, but one could just as well use a gravitational potential. We will study in detail the bound states of the hydrogen atom according to the old quantum theory

[1] We define $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$.

and, in particular, derive the energy associated with a given trajectory of the electron of mass m_1 around the nucleus of mass m_2.

3.1.1 Center of mass

Let $\mathbf{r}_1, \mathbf{r}_2$ be the position vectors of the electron and the nucleus relative to an arbitrary reference frame, and $\mathbf{v}_1, \mathbf{v}_2$ their respective velocities.

1. Write the Lagrangian $\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2)$.
2. Let $\mathbf{R}$ be the center-of-mass position vector and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Show that the Lagrangian can be written as:
$$\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2) = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v}) \tag{3.1}$$
3. Explain why the total angular momentum about the center of mass G, denoted $\mathbf{J}$, is a conserved quantity. Deduce a conclusion about the trajectory.

From here on, we examine only the internal motion through $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to $\mathbf{J}$.

3.1.2 Integration of the equations of motion

1. Write the Hamiltonian $\mathcal{H}$ for the internal motion and derive Hamilton's equations. Recover the conservation of angular momentum and interpret the equation involving only $\mathbf{r}, \dot{\mathbf{r}}$.
2. Determine the relationship between r and θ, i.e., the trajectory. To do so, eliminate time from the previous equations by setting $u = \frac{1}{r}$, and show that:
$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = K, \quad K = \frac{\mu\vartheta^2}{J^2} \tag{3.2}$$
3. Finally, deduce that the trajectory is a conic, whose equation can always be written in the form:
$$r(\theta) = \frac{p}{1 + \varepsilon\cos\theta} \tag{3.3}$$
Give the expression of p, the conic parameter, and ε, the eccentricity. Check how the value of ε relative to 1 determines the nature of the corresponding state (bound or unbound).

3.1.3 Bohr quantization

In this part, we consider only bound states ($E < 0$) and apply Bohr's rules to select among all classically possible trajectories. These rules involve the action variables J_θ, J_r and are written:

$$J_\theta := \oint p_\theta \,\mathrm{d}\theta = n_\theta h \tag{3.4}$$

$$J_r := \oint p_r \,\mathrm{d}r = n_r h \tag{3.5}$$

$$n_\theta, n_r \in \mathbb{Z} \tag{3.6}$$

1. Determine the possible values of the angular momentum J as a consequence of the quantization of J_θ. Specify the possible values of the integer n_θ.

2. Quantize J_r and deduce the relation between ε and the integers n_r, n_θ[2]. Given:

$$\int_0^{2\pi} \frac{1}{1+\varepsilon\cos\theta}\,\mathrm{d}\theta = \frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{3.7}$$

3. Deduce that the energy E is quantized, with $n \in \mathbb{N}^*$ depending on n_θ, n_r, and that:

$$E_n = -\frac{\mu\vartheta^4}{2n^2\hbar^2} \tag{3.8}$$

[2] At first glance, one might say that $J_r = 0$; an integration by parts is necessary.

3.2 Rutherford Scattering Cross Section [4] (SP) ★★

(Solution)

We consider the same situation as in the previous exercise: two particles, one of which is fixed, interacting through a potential of the form $V(r) = \frac{C}{r}$. Here, $C = \frac{Qq}{4\pi\varepsilon_0}, Q = Ze, q = 2e$. We will use some results from the previous exercise, so it is recommended to complete that one first.

3.2.1 Deflection of a charged particle by an atomic nucleus

We work in the polar coordinate system (r, φ), perpendicular to the angular momentum, since the motion is planar. The alpha particle arrives with initial velocity $\mathbf{v}_0$. We assume $\lim_{t\to-\infty} \varphi(t) = \pi$.

1. Determine the non-zero component of $\mathbf{J}$ as a function of r, φ. Also determine it in terms of b, v_0, where b is the impact parameter.

2. Write the equation of motion. Decompose $\mathbf{v} = \dot{\mathbf{r}}$ into a vector parallel and one perpendicular to the polar axis. Deduce that:
$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi \tag{3.9}$$

3. We want to introduce the deflection angle θ. By integrating the equation, show that:
$$v_0 \sin\theta = \frac{C}{mbv_0}(\cos\theta + 1) \tag{3.10}$$

4. Using some trigonometric identities, deduce that:
$$\tan\frac{\theta}{2} = \frac{C}{2E_0 b} \tag{3.11}$$
where $E_0 = \frac{1}{2}mv_0^2$.

3.2.2 Rutherford Scattering Cross Section

1. Recall the formula for the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

2. Deduce that:
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{C^2}{16E_0^2\sin^4\frac{\theta}{2}} \tag{3.12}$$

3. Deduce that this model is invalid for small deflection angles.

4. Explain why this experiment demonstrates the existence of atomic nucleus.

3.3 Cherenkov Effect [5] (SR, NP) ★★★

(Solution) The Cherenkov effect occurs when a charged particle travels through a dielectric medium at a speed v greater than the speed of light in that medium c/n, where n is the refractive index of the medium.

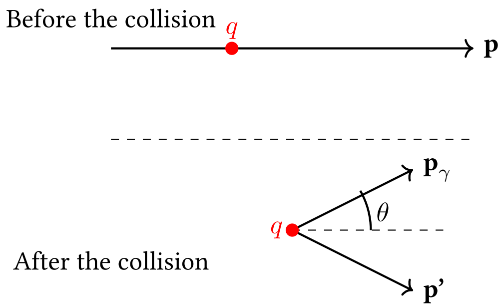

Figure 3.2: Diagram of the Cherenkov effect.

The momentum of the charged particle is $\mathbf{p}$ before the collision, $\mathbf{p}_\gamma$ is the photon momentum after the collision, and $\mathbf{p}'$ is the particle's momentum after the collision, (c.f fig 3.2). The angle θ is the angle between $\mathbf{p}$ and $\mathbf{p}_\gamma$. Recall that $\lambda = \frac{c}{n\nu}$.

1. Express p_γ in terms of h, ν, c, n. Deduce the relation between p_γ and E_γ in a medium with refractive index n.
2. Write the momentum conservation equation for the elementary process.
3. Using the previous question, express $\mathbf{p}'^2$ in terms of the magnitudes of the momenta p, p_γ, and the angle θ.
4. Write the energy conservation equation.
5. E is the initial energy of the electron. Deduce that:

$$p'^2 = p^2 - 2\frac{E}{c^2}h\nu + \frac{p_\gamma^2}{n^2} \tag{3.13}$$

6. Express $\cos\theta$ in terms of $p, p_\gamma, E, h, n, c, \nu$.
7. Show that:

$$\cos\theta = \frac{c}{nv}\left[1 + \frac{h\nu}{2E}(n^2 - 1)\right] \tag{3.14}$$

8. What is the condition for the Cherenkov effect to occur?
9. In which frequency range are the photons emitted?
10. In which direction are the highest-energy photons emitted?
11. All photons are emitted within a cone; what is the half-apex angle ϕ of this cone? Estimate ϕ for $n = \frac{4}{3}$ and $v = \frac{4}{5}c$.
12. Compare the minimum kinetic energy required for the particle to produce Cherenkov radiation in the cases of an electron and a proton, for $n = \frac{4}{3}$.

3.4 Pulsed Magnetic Field Machine (EM) ★★

(Solution)

The magnetic stimulation machine is a non-invasive technology used in physiotherapy and rehabilitation. It works by generating pulsed magnetic fields using a circular coil. In practice, the machine sends current pulses through the coil, which creates a time-varying magnetic field. According to Faraday's law, this variation automatically induces an electric field in the surrounding tissues.

This induced electric field acts directly on the cellular membranes of muscles by activating ion channels. As a result, an action potential is triggered, leading to muscle contraction. This mechanism allows not only for the stimulation of weakened or atrophied muscles, but also improves blood circulation and reduces pain. Moreover, the absence of direct skin contact makes the treatment comfortable and safe for the patient.

To modelize this phenomen, we consider a circular coil of radius R carrying a time-varying current:

$$I(t) = I_0 e^{-t/\tau} \sin(\omega t), \tag{3.15}$$

where I_0 is the current amplitude, τ is the damping time constant, and ω is the oscillation frequency. The coil's axis is assumed to coincide with the z-axis. The coil is considered thin and modeled as a single loop.

1. **Magnetic field of the coil**
   (a) Assuming the coil behaves like a magnetic dipole, express the magnetic field $\mathbf{B}$ along the central axis (at a distance z from the center) in terms of $I(t)$, R, z, and physical constants.
   (b) Show that for $z \gg R$, the field approximates that of a magnetic dipole and give its asymptotic expression.
2. **Induced electric field in biological tissue** We model the tissue as a thin conducting disk of radius a, placed under the coil.
   (a) Starting from the local Faraday law:
   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.16}$$
   express the induced electric field $\mathbf{E}$ in terms of $\frac{\mathrm{d}B}{\mathrm{d}t}$.
   (b) Assuming cylindrical symmetry (purely azimuthal field), derive the expression for the induced electric field $E_\theta(r,t)$ in the plane of the disk, distinguishing the cases $r < R$ and $r > R$.
3. **Effect on motor neurons** A motor neuron is assumed to be activated when the induced voltage exceeds a threshold V_{thresh}.
   (a) Express V in terms of the parameters of the problem.
   (b) Determine a condition on I_0, τ, ω, and the geometric parameters to ensure neuron activation.
4. **Effect of pulsed magnetic field on muscles** Explain why a pulsed magnetic field, by inducing an electric field in tissues, can provoke muscle contraction. Briefly describe the physiological mechanism (activation of ion channels, generation of an action potential, muscle contraction).

3.5 Metric on a Sphere (AM) ★★★

(Solution)

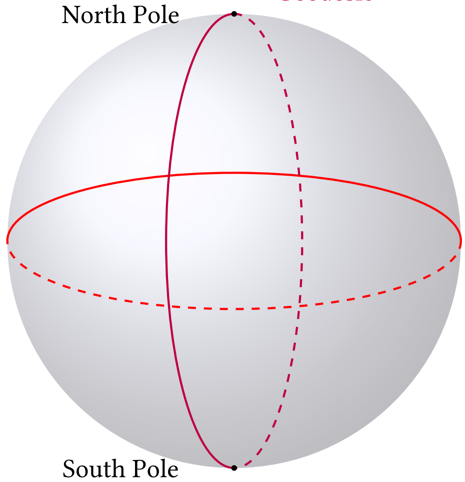

Figure 3.3: Diagram of a sphere and its geodesics.

Our goal is to determine the metric on a sphere and its geodesics. This will help us understand the optimal flight paths for an airplane. Recall that in spherical coordinates, for a fixed radius R,

$$x = R\cos\varphi\sin\theta \tag{3.17}$$

$$y = R\sin\varphi\sin\theta \tag{3.18}$$

$$z = R\cos\theta \tag{3.19}$$

1. Calculate the line element $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ as a function of R, θ, and φ.
2. Using the action $S = \int \mathrm{d}s = \int \mathcal{L}\mathrm{d}\lambda$, where λ is a suitably chosen parameter, and the variational principle, determine the geodesic equations.
3. Solve the equations by using symmetries. One may use that

$$\int \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1 - \frac{\lambda^2}{\sin^2\alpha}}} \quad \text{set } u = \cot\alpha, \tag{3.20}$$

$$\int -\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arccos t + C. \tag{3.21}$$

Show that the geodesics have the following form:

$$(x, y, z) \in S^2, \quad ax + by + cz = 0, \tag{3.22}$$

that is, the geodesics are intersections between planes passing through the origin and the sphere, or in other words, arcs of great circles.

3.6 Blackbody Radiation (PS) ★★★★

(Solution)

We seek to obtain the spectral energy density, that is the function

$$u(\nu, T) = \frac{\mathrm{d}^2 W}{\mathrm{d}\nu \mathrm{d}\mathcal{V}} = \frac{\mathrm{d}N}{\mathrm{d}\nu} \frac{\langle W \rangle}{\mathcal{V}}, \tag{3.23}$$

with W the energy and $\langle W \rangle$ the mean energy. We will also work in a historical framework, without using quantum mechanics, which was partly discovered thanks to the results we are about to demonstrate.

3.6.1 Number of Modes Excited per Frequency Unit

1. Consider a blackbody represented by a cubic cavity of side length L and volume $\mathcal{V}$. Write down the wave equation for the electric field $\mathbf{E}$ inside the cavity.

2. Solve the wave equation. Explain why the field $\mathbf{E}$ depends on three modes $n_x, n_y, n_z \in \mathbb{N}^*$.

3. Show that

$$n_x^2 + n_y^2 + n_z^2 = r^2 = \left(\frac{2L}{\lambda}\right)^2. \tag{3.24}$$

4. By counting unit cubes stacked along the axes n_x, n_y, n_z, we can enumerate the total number N of excited modes.

   Each cube can be represented as $\mathbf{r} = n_\mu \mathbf{e}^\mu$. When the cubes are very numerous, that is, when L is much larger than λ, it suffices to calculate the volume of a sphere of radius r.

   However, since the integers are strictly positive, only $1/8$ of the total sphere volume is taken. Also, a factor of 2 must be considered due to the two possible polarization planes of the electric field $\mathbf{E}$.

   Using these data, deduce that

$$\frac{\mathrm{d}N}{\mathrm{d}\nu} = \frac{8\pi\nu^2}{c^3}\mathcal{V}. \tag{3.25}$$

3.6.2 Ultraviolet Catastrophe

1. Explain why the ensemble associated with this problem — the calculation of u — corresponds to the canonical ensemble.

2. Calculate the Hamiltonian of a harmonic oscillator.

3. Give the probability of being in an energy state W. Deduce the partition function Z of a gas of harmonic oscillators.

4. Show that

$$\langle W \rangle = k_B T. \tag{3.26}$$

5. Deduce that

$$u(\nu, T) = 8\pi \frac{\nu^2}{c^3} k_B T, \tag{3.27}$$

   and explain the title of this subsection.

3.6.3 Planck's Law

The revolutionary idea is to estimate that the photon energy is quantized. Thus, we move from the idea of a continuous energy distribution to a discrete one. This idea arose from the fact that the average energy of an oscillator did not depend on the frequency ν. Planck suspected a simple proportionality relation between W and ν:

$$W_n = nh\nu. \tag{3.28}$$

Then came the idea of quanta, that energy is not continuous but distributed in indivisible packets called **quanta**[3].

1. Recalculate the partition function Z.
2. Deduce that
$$u(\nu, T) = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{\beta h\nu} - 1} \tag{3.29}$$
with $\beta = \frac{1}{k_B T}$.

Thus, the ultraviolet catastrophe was resolved, and this result agreed perfectly with experiments. This function became integrable, which later led to Stefan's law.

3.6.4 Energy Flux Emitted by a Blackbody

Consider a cavity in thermal equilibrium filled with a photon gas at temperature T. The radiation is **isotropic** and characterized by a volumetric spectral energy density $u(\nu)$, such that

$$u(\nu)\,\mathrm{d}\nu = \text{electromagnetic energy per unit volume between frequencies } \nu \text{ and } \nu + \mathrm{d}\nu. \tag{3.30}$$

Let I be the total intensity (energy flux per unit surface perpendicular to it, integrated over all directions) emitted by the blackbody.

1. Recall the expression of the monochromatic energy flux emitted in a direction making an angle θ with respect to the surface normal, in terms of the directional spectral intensity I_ν and the solid angle $\mathrm{d}\Omega$.
2. Show that the total energy flux emitted at frequency ν per unit surface is given by
$$I(\nu) = \int_{\Omega_+} I_\nu \cos\theta \,\mathrm{d}\Omega, \tag{3.31}$$
where Ω_+ denotes the outgoing hemisphere $(0 \leq \theta \leq \pi/2)$.
3. Assuming the radiation is isotropic, i.e., I_ν is independent of direction, show that
$$I(\nu) = \pi I_\nu. \tag{3.32}$$
4. By integrating over all frequencies, deduce that the total emitted intensity is
$$I = \int_0^\infty \pi I_\nu \,\mathrm{d}\nu. \tag{3.33}$$

[3] Albert Einstein used Planck's idea in his annus mirabilis of 1905 to explain the photoelectric effect, which earned him the Nobel Prize in 1921.

5. Show that the volumetric spectral energy density $u(\nu)$ is given by

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \, \mathrm{d}\Omega. \tag{3.34}$$

Assuming isotropic radiation, deduce that

$$u(\nu) = \frac{4\pi}{c} I_\nu. \tag{3.35}$$

6. Deduce that

$$I = \frac{c}{4} \int_0^\infty u(\nu) \, \mathrm{d}\nu. \tag{3.36}$$

3.6.5 Stefan's Law

Stefan's law states that for a blackbody,

$$I(T) = \sigma T^4, \tag{3.37}$$

where σ is the Stefan–Boltzmann constant. We will prove it.

1. Using the previous parts, show that

$$I = \frac{2\pi k_B^4}{h^3 c^2} T^4 \int_0^\infty \frac{x^3}{e^x - 1} \mathrm{d}x. \tag{3.38}$$

2. Verify the convergence of the integral and express it as a series.
3. Finally, prove that

$$I(T) = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4. \tag{3.39}$$

This is recognized as Stefan's law[4]

$$I = \sigma T^4. \tag{3.40}$$

3.6.6 Application: Solar Mass Loss by Electromagnetic Radiation

Assuming the Sun is a blackbody, determine $\dot{m}$, the mass loss per unit time. What is this mass loss rate in $\mathrm{kg} \cdot \mathrm{s}^{-1}$? Knowing that our Sun is approximately 4.6×10^9 years old, estimate how many Earth masses the Sun has lost so far.

Data: $R = 6.96 \times 10^8\,\mathrm{m}$, $T = 5775\,\mathrm{K}$, $m = 1.98 \times 10^{30}\,\mathrm{kg}$, $m_\oplus = 6 \times 10^{24}\,\mathrm{kg}$.

[4] Hence, $\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$, which is rather unexpected.

3.7 Minimization of the Gravitational Potential by a Ball (AM) ★★★★★

(Solution)

This exercise involves notions of differential calculus.

We consider the following variational problem: among bounded open domains $\Omega \subset \mathbb{R}^3$ of fixed volume, find the one minimizing the **internal gravitational interaction** defined by the functional:

$$\mathcal{F}[\Omega] = \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x' \tag{3.41}$$

Note that this expression is proportional to the gravitational self-interaction potential of a body with uniform density. Indeed, for $x \in \mathbb{R}^3$,

$$U(x) = -G\int_{\Omega} \frac{\rho}{|x-x'|}\,\mathrm{d}^3x' \tag{3.42}$$

The total gravitational potential energy of the system is then:

$$E[\Omega] = \frac{1}{2}\int_{\Omega} \rho U(x)\,\mathrm{d}^3x = -\frac{G}{2}\rho^2\int_{\Omega}\int_{\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'. \tag{3.43}$$

- We consider a domain $\Omega \subset \mathbb{R}^3$, i.e., a bounded open set of class $\mathcal{C}^2$, with boundary $\partial\Omega$.
- The volume of Ω is defined by:
$$V := \int_{\Omega} \mathrm{d}^3x \tag{3.44}$$
- We consider an infinitesimal normal deformation of the boundary of Ω, parametrized by $\varepsilon \in \mathbb{R}$, given by:
$$x \mapsto x + \varepsilon f(x) n(x), \quad \text{for } x \in \partial\Omega \tag{3.45}$$
where $f \in C^\infty(\partial\Omega)$ is a smooth function and $n(x)$ is the outward unit normal vector to $\partial\Omega$.
- The deformed domain is denoted Ω_ε, the bounded open set obtained by this deformation:
$$\Omega_\varepsilon := \{x + \varepsilon f(x) n(x) \mid x \in \Omega\} + o(\varepsilon) \tag{3.46}$$
(The deformation is assumed to be smoothly extended inside Ω to rigorously define Ω_ε.)

3.7.1 Hadamard's Formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and Ω_ε a smooth deformation of Ω such that for $x \in \partial\Omega$,

$$x \mapsto x + \varepsilon f(x) n(x) \tag{3.47}$$

and assume this deformation extends smoothly to all of Ω.

We want to prove that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\,\mathrm{d}^3x = \int_{\partial\Omega} F(x) f(x)\,\mathrm{d}S(x) \tag{3.48}$$

where $\mathrm{d}S$ is the surface element associated to $\partial\Omega$.

1. We will study the function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.

   (a) Justify that $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, M \mapsto \det M$ is differentiable.

   (b) Prove that for all $M \in \mathcal{M}_n(\mathbb{R})$,

   $$\det(I + \varepsilon M) = 1 + \varepsilon \operatorname{Tr}(M) + o(\varepsilon) \tag{3.49}$$

   Deduce that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \det(I + \varepsilon M)\big|_{\varepsilon=0} = \operatorname{Tr}(M)$.

   (c) Let $X \in \mathrm{GL}_n(\mathbb{R})$, $H \in \mathcal{M}_n(\mathbb{R})$. Prove that

   $$\mathrm{d}(\det)(X)(H) = \operatorname{Tr}({}^t \operatorname{Com}(X)\, H) \tag{3.50}$$

2. Set the change of variables $x(u) = u + \varepsilon f(u) n(u)$, and compute the Jacobian $\det\left(\frac{\partial x}{\partial u}\right)$ at first order in ε, i.e., up to $o(\varepsilon)$.

3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$, and $\varepsilon \in U$ a neighborhood of 0. By considering a well-chosen function, prove that for all $x \in \mathbb{R}^n$,

   $$F(x + \varepsilon v(x)) = F(x) + \varepsilon v(x) \cdot \nabla F(x) + o(\varepsilon) \tag{3.51}$$

4. Deduce the desired result using the Divergence Theorem.

3.7.2 Connection with the Gravitational Potential

1. Show that $E[\Omega] < 0$, and that minimizing the energy is equivalent to maximizing the following quantity:

   $$\mathcal{I}[\Omega] := \int_\Omega \int_\Omega \frac{1}{|x - x'|} \,\mathrm{d}^3x \,\mathrm{d}^3x'. \tag{3.52}$$

2. Suppose $\Omega = B(0, R)$ is a ball centered at the origin of radius R such that $\mathrm{Vol}(\Omega) = \frac{4}{3}\pi R^3 = V$. Show that the gravitational potential at the center is given by:

   $$U(0) = -G\rho \int_\Omega \frac{1}{|x'|} \,\mathrm{d}^3x'. \tag{3.53}$$

   Calculate this integral explicitly.

3.7.3 The Sphere?

1. Prove that

   $$\delta\mathcal{F} = 2 \int_{\partial\Omega} \left(\int_\Omega \frac{1}{|x - x'|} \,\mathrm{d}^3x' \right) f(x) \,\mathrm{d}S(x) \tag{3.54}$$

   You may use or prove (for the more courageous) that for all $\Omega \subset \mathbb{R}^n$, for all $\varphi : \Omega \to \mathbb{R}^n$,

   $$\int_{\partial(\Omega^2)} \varphi(x) \,\mathrm{d}\mu(x) = 2 \int_{\Omega \times \partial\Omega} \varphi(x) \,\mathrm{d}\mu(x) \tag{3.55}$$

2. We want to minimize $\mathcal{F}$ under fixed volume constraint V. To do this, consider the Lagrangian:

   $$\mathcal{L}(\lambda) = \mathcal{F} - \lambda V, \quad \lambda \in \mathbb{R}. \tag{3.56}$$

   Deduce that the first variation of $\mathcal{F}$ writes:

   $$\delta\mathcal{L} = \int_{\partial\Omega} \left(2 \int_\Omega \frac{1}{|x - x'|} \,\mathrm{d}^3x' - \lambda \right) f(x) \,\mathrm{d}S(x). \tag{3.57}$$

3. Using spherical symmetry, show that if Ω is a ball of radius R, then for all $x \in \partial\Omega$, the quantity
$$\int_\Omega \frac{1}{|x - x'|} \, \mathrm{d}^3 x' \tag{3.58}$$
is constant. Deduce that the ball satisfies the **stationary condition** $\delta\mathcal{L} = 0$ for all f.

4. (*Bonus*) Show that the ball is indeed a *local minimum* for $\mathcal{F}$ under volume constraint by studying the second variation.

5. Conclude and explain why large objects in the Universe are spherical.

3.8 Relativistic Motion of a Charged Particle (SR, AM, EM, SM) ★★★★★

(Solution)

3.8.1 Relativistic Lagrangian of a Charged Particle in an Electromagnetic Field

1. Show that using the principle of least action and Lorentz invariance, the action of a free particle of mass m can be written as $S = -mc\int \mathrm{d}s$ where $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2$. Deduce that the Lagrangian of the system is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}\,, \tag{3.59}$$

   where $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$.

2. By introducing the electromagnetic four-potential $A^\mu = (\phi/c, \mathbf{A})$, propose an interaction term L_{int} corresponding to a particle of charge q in this field. Show that it can be written as

$$\mathcal{L}_{\text{int}} = q\,\mathbf{A}\cdot\mathbf{v} - q\phi\,, \tag{3.60}$$

   and deduce the total Lagrangian $\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{int}}$.

3. Starting from the total Lagrangian, calculate the generalized momentum $P_i = \partial\mathcal{L}_{\text{tot}}/\partial v^i$. Show that it can be expressed as

$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A}\,, \tag{3.61}$$

   where $\gamma = (1 - v^2/c^2)^{-1/2}$.

4. Write the Euler–Lagrange equations associated with L_{tot} and show that they lead to the Lorentz equation in 3 dimensions,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)\,, \tag{3.62}$$

   with $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$ and, $\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})$.

5. Express the Lagrangian by parameterizing with proper time τ and deduce that,

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + qg_{\mu\nu}A^\mu\dot{x}^\nu \tag{3.63}$$

6. Show that

$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu \tag{3.64}$$

   Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

7. Explicitly write the components of the tensor $F_{\mu\nu}$ and show that $F_{0i} = E_i/c$ and $F_{ij} = -\varepsilon_{ijk}B_k$. Interpret the physical meaning of these components.

8. Calculate the two invariants of the electromagnetic field,

$$I_1 = F_{\mu\nu}F^{\mu\nu}, \quad I_2 = \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{3.65}$$

   and express them in terms of $\mathbf{E}$ and $\mathbf{B}$. What are the physical cases corresponding to $I_1 = 0$ and $I_2 = 0$?

9. Verify that under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, the equations of motion remain unchanged. What is the associated symmetry?

3.8.2 Equations of Motion of a Charged Particle in a Plane Electromagnetic Wave

Consider a particle of mass m and charge q subjected to an electromagnetic field described by the tensor $F^{\mu\nu}$. Its motion is governed by the equation:

$$m\ddot{x}^{\mu} = qF^{\mu\nu}\dot{x}_{\nu} \tag{3.66}$$

where the dots denote derivatives with respect to the particle's proper time τ. We use natural units where $c = 1$.

A plane electromagnetic wave is modeled by a four-potential of the form:

$$A^{\mu}(x) = a^{\mu} f(k_{\nu}x^{\nu}) \tag{3.67}$$

where f is a $\mathcal{C}^1$ function, k^{μ} is a lightlike four-vector, hence $k^{\mu}k_{\mu} = 0$, and a^{μ} is a constant four-vector representing the polarization.

1. Show that
$$F^{\mu\nu}(x) = (k^{\mu}a^{\nu} - k^{\nu}a^{\mu})\, f'(k_{\rho}x^{\rho}) \tag{3.68}$$
2. (a) Calculate $\partial_{\mu}A^{\mu}$ for the potential $A^{\mu}(x) = a^{\mu} f(k_{\rho}x^{\rho})$.
   (b) Deduce that the Lorenz gauge condition $\partial_{\mu}A^{\mu} = 0$ implies:
$$a^{\mu}k_{\mu} = 0 \tag{3.69}$$
3. Now consider the motion of a particle in this electromagnetic wave.
   (a) Using the expression for the tensor $F^{\mu\nu}$ found in question 1, show that:
$$F^{\mu\nu}\dot{x}_{\nu} = [k^{\mu}(a_{\rho}\dot{x}^{\rho}) - a^{\mu}(k_{\rho}\dot{x}^{\rho})]\, f'(k_{\rho}x^{\rho}) \tag{3.70}$$
   (b) Deduce the equation of motion in the form:
$$m\ddot{x}^{\mu} = q\,[k^{\mu}(a_{\rho}\dot{x}^{\rho}) - a^{\mu}(k_{\rho}\dot{x}^{\rho})]\, f'(k_{\rho}x^{\rho}) \tag{3.71}$$
4. Now we seek to integrate this equation.
   (a) Show that the scalar $k_{\rho}\dot{x}^{\rho}$ is constant during the motion.
   (b) Deduce that $\phi = k_{\rho}x^{\rho}(\tau)$ is an affine function of τ, which can be used as a new parameter.
   (c) Using the previous relations, integrate the equation of motion and determine the complete expression for the trajectory $\tau \mapsto x^{\mu}(\tau)$[5]

[5] This exercise allows us to analytically determine the trajectory of a charged particle in a plane electromagnetic wave. You can then plot it in Python using the obtained functions.

3.8.3 Field Theory

We define the action,

$$S = \int_\Omega -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A^\mu j_\mu \mathrm{d}^4 x, \quad \Omega \subset \mathbb{R}^{1,3} \tag{3.72}$$

We can thus easily define a Lagrangian **density**.

1. For an action depending on a field φ (scalar, tensor, etc.):

$$S = \int_\Omega \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu) \mathrm{d}^4 x \tag{3.73}$$

   Prove that the Euler-Lagrange equations remain valid for a field φ.
   To do this, we will postulate the principle of least action, meaning that for an infinitesimal transformation $\varphi \mapsto \varphi + \varepsilon\eta$[6], we have,

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}[\varphi + \varepsilon\eta, \partial_\mu(\varphi + \varepsilon\eta), x^\mu](0) = 0 \tag{3.74}$$

2. Derive Maxwell's equations in tensor form,

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,, \tag{3.75}$$

   where $j^\mu = (c\rho, \mathbf{j})$ is the four-current (four-current density).

3.8.4 Trajectory of a Charged Particle in a Constant Magnetic Field

Consider a particle of mass m and charge q moving relativistically in an electromagnetic field. In this section, we gradually introduce the effects of a constant magnetic field $\mathbf{B} = B\,\mathbf{e}_z$ (curved sector of a synchrotron) and an average braking force due to synchrotron radiation.

A. Synchrotron Radiation Neglected

1. Calculate $F^{\mu\nu}$.

2. Deduce that the motion is in the Oxy plane. Show that the energy is constant if radiation losses are neglected.

3. Show that, in the absence of energy loss, for $u^\mu = (\gamma c, 0, u_0 = \gamma v, 0)$[7],

$$x(t) = R\cos\left(\frac{\omega}{\gamma}t\right), \qquad y(t) = R\sin\left(\frac{\omega}{\gamma}t\right) \tag{3.76}$$

   With (synchrotron law)[8]:

$$R = \frac{\gamma v}{\omega} = \frac{\gamma m v}{qB} \tag{3.77}$$

[6] Where η is a $\mathcal{C}^1(\Omega)$ function, and $\forall x \in \partial\Omega, \eta(x) = 0$, i.e., the function vanishes at the boundaries.
[7] It would also be necessary to show that γ and v are constant and that $\tau(t) = \gamma t$.
[8] For this, we will need to switch to the laboratory frame.

B. Study of the Real Motion

1. Synchrotron radiation leads to an average energy loss. Recall the formula for the average radiated power (relativistic Larmor) for a centripetal acceleration $a = v^2/R$,

$$P = -\frac{\mathrm{d}}{\mathrm{d}t}E \;=\; \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^4 a^2 \tag{3.78}$$

   Using $E = \gamma mc^2$, show that by expanding, we obtain the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -C\,(\gamma^2 - 1) \tag{3.79}$$

   Give the expression for the coefficient C in terms of $q, B, m, c, \varepsilon_0$.

2. Solve the differential equation for γ[9]. We give, $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$.
3. Deduce the new trajectory of the charged particle. Study the limit as $t \to \infty$.
4. Plot the parametric curve $x(t), y(t)$ in Python. What problem does this generate?

3.8.5 Physics of Relativistic Colliders

Here, we will use natural units where the speed of light $c = 1$.

1. Define the square of the total energy-momentum invariant $s = (p_1 + p_2)^2$ for the collision of two particles with four-momenta p_1 and p_2. Express the total energy available in the center-of-mass frame (CMS) in terms of s.
2. For a head-on collision of two identical particles of mass m and energy E (each) in the laboratory frame, show that the CMS energy is $\sqrt{s} = 2E$ (assuming $E \gg m$).
3. For the case of a collision with a fixed target of mass m, derive the formula

$$s = m^2 + m^2 + 2mE_{\mathrm{lab}}, \tag{3.80}$$

   and deduce the threshold energy for the production of two particles of mass m (extreme elastic collision).

4. Calculate the energy required in a fixed-target experiment to produce a new particle of mass M at threshold, and compare it to the energy required in a symmetric collider ($E_{\mathrm{CM}} = M + M$). Why are colliders with counter-propagating beams more efficient for reaching high energies?

[9] It is indeed much simpler to solve the equation for γ than for v, since here v depends on time.

3.9 Relativistic Hydrodynamics and Heavy-Ion Collisions (SR,SM) ★★★★★

(Solution)

We work in natural units with $c = 1$.

3.9.1 Classical Hydrodynamics

1. Write the mass conservation (continuity) equation for a classical fluid, namely
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.81}$$
Show that in the case of an incompressible fluid ($\rho =$ constant), this reduces to $\nabla \cdot \mathbf{v} = 0$.

2. Write Euler's equation for a perfect (non-viscous) fluid under the influence of a gravitational field $\mathbf{g}$:
$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \rho \mathbf{g}. \tag{3.82}$$
Briefly describe the physical meaning of each term in this equation.

3. Show how the inclusion of viscous effects leads to the Navier–Stokes equation:
$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \eta \nabla^2 \mathbf{v} + \Big(\zeta + \frac{\eta}{3}\Big)\nabla(\nabla \cdot \mathbf{v}) + \rho \mathbf{g}, \tag{3.83}$$
where η is the shear (dynamic) viscosity and ζ is the bulk viscosity. Explain the role of these terms.

4. Explain the difference between the Lagrangian description (tracking fluid particle trajectories) and the Eulerian description (observing the velocity field at a fixed point in space). In particular, show that the total derivative for a fluid is $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ in the Eulerian formalism.

5. Define streamlines in a fluid, and show that these curves are tangent to the velocity vector field $\mathbf{v}$ at each point. Interpret these lines physically.

6. Derive Bernoulli's theorem for a stationary, incompressible, and non-viscous fluid. Show that along a streamline,
$$\frac{1}{2}\rho v^2 + p + \rho \Phi = \text{constant}, \tag{3.84}$$
where Φ is a potential of forces (e.g. $\Phi = gz$ in a constant gravitational field $\mathbf{g}$).

3.9.2 Introduction to Relativistic Hydrodynamics

Relativistic hydrodynamics describes the evolution of continuous systems with high energy density (such as the quark–gluon plasma) incorporating the principles of special relativity. We consider here perfect fluids, without viscosity or heat conduction, and their contravariant description.

1. **Energy–momentum tensor.** The energy and dynamical content of a perfect fluid is encoded in the energy–momentum tensor:
$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu}, \tag{3.85}$$
where:

- ε is the energy density (in the fluid's rest frame),
- p is the pressure (same unit as ε, i.e. J/m^3),
- u^μ is the fluid four-velocity,
- $\eta^{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric.

(a) Verify that $T^{\mu\nu}$ is symmetric.

(b) Compute $T^{\mu\nu}$ in the fluid's rest frame ($u^\mu = (1,0,0,0)$).

(c) Interpret the physical components T^{00}, T^{0i}, and T^{ij}.

(d) Calculate the trace $T^\mu_{\ \mu}$.

(e) Show that in the ultra-relativistic gas case, $p = \frac{\varepsilon}{3}$.
Recall from statistical physics,

$$\varepsilon = \frac{1}{V}\int \frac{\mathrm{d}^3\|\mathbf{p}\|}{(2\pi)^3} E(\mathbf{p}) \times f(\mathbf{p}) \tag{3.86}$$

$$p = \frac{1}{3V}\int \frac{\mathrm{d}^3\|\mathbf{p}\|}{(2\pi)^3} \frac{\|\mathbf{p}\|^2}{E(\mathbf{p})} \times f(\mathbf{p}). \tag{3.87}$$

2. **Conservation of energy and momentum.** In any isolated system, the energy–momentum tensor is locally conserved:

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.88}$$

This tensorial equation (4 scalar equations) expresses the conservation of energy ($\nu = 0$) and of the three momentum components ($\nu = 1,2,3$). It constitutes the fundamental equation of relativistic hydrodynamics.

(a) What are the dynamical unknowns of the problem?

(b) Why must this system be supplemented by an equation of state relating ε, p, and possibly T?

3. **Relativistic thermodynamics.** In the fluid's rest frame, we locally define:

T : temperature, s : entropy density, μ : chemical potential, n : particle density.

The first law of thermodynamics, expressed in local densities (i.e. in a volume element $\mathrm{d}V$), takes the form:

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}n. \tag{3.89}$$

Using the classical first law,

$$\mathrm{d}U = T\,\mathrm{d}S + \mu\,\mathrm{d}N - p\mathrm{d}V \tag{3.90}$$

and assuming $\mu = 0$, prove the identity $\varepsilon + p = Ts$, called Euler's relation.

4. **Relativistic speed of sound.** The speed of sound c_s is defined by:

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_s. \tag{3.91}$$

(a) Compute c_s for an ultra-relativistic fluid where $p = \varepsilon/3$.

(b) Compare to the speed of light $c = 1$ and comment.

3.9.3 Relativistic equation of motion

We consider a perfect fluid in special relativity. The total number of particles is given by

$$N = \int_\Sigma j^\mu \, \mathrm{d}\Sigma_\mu, \quad j^\mu = nu^\mu \tag{3.92}$$

over a future-oriented spacelike hypersurface Σ (for example $t = \text{const}$). We assume that N is conserved.

1. Show that particle number conservation is locally expressed as

$$\partial_\mu(nu^\mu) = 0, \tag{3.93}$$

   where n is the particle density in the comoving frame, and u^μ the fluid four-velocity.

2. Using $\partial_\mu T^{\mu\nu} = 0$, deduce the equation of motion (or *relativistic Euler equation*) of a perfect fluid without sources and isotropic particle density:

$$(\varepsilon + p)u^\mu \partial_\mu u^\nu + (u^\mu u^\nu + g^{\mu\nu})\, \partial_\mu p = 0. \tag{3.94}$$

3.9.4 Application to heavy-ion collisions

We introduce the Bjorken coordinates: $\tau = \sqrt{t^2 - z^2}$, $\eta = \frac{1}{2} \ln \frac{t+z}{t-z}$.

1. Compute the line element $\mathrm{d}s^2$, and deduce $g_{\mu\nu}$.

2. Assuming a boost-invariant fluid along z, we are no longer in flat space. The conservation equation then reads,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0 \tag{3.95}$$

   where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right). \tag{3.96}$$

   Show that the conservation equation leads to:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{\varepsilon + p}{\tau} = 0. \tag{3.97}$$

3. For $p = \varepsilon/3$, solve the above equation and deduce:

$$\varepsilon(\tau) \propto \tau^{-4/3}, \quad T(\tau) \propto \tau^{-1/3}. \tag{3.98}$$

4. During the QGP $\to$ hadron transition, the equation of state can be written as:

$$p = \frac{\varepsilon - 4B}{3}. \tag{3.99}$$

   Show that $p = 0$ at the transition implies $\varepsilon = 4B$ and deduce the critical temperature T_c.

5. By modelling a nucleus as a sphere of radius R, define the geometric cross-section $\sigma \simeq \pi(2R)^2$. Relate this quantity to the distinction between central and peripheral collisions.

6. Show that the initial energy density ε_0 is larger for a central collision. Assuming $\varepsilon = aT^4$, estimate the initial temperature T_0 reached at RHIC ($\varepsilon_0 \sim 10$ GeV/fm^3).

3.10 Hydrogen Atom and Radial Equation [6] (QM) ★★★

(Solution)

In this problem, we study the hydrogen atom (an electron of mass m_e in the Coulomb potential $V(r) = -e^2/r$ of a fixed proton) in non-relativistic quantum mechanics. We use spherical coordinates (r, θ, ϕ) and the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}\right]\psi(r,\theta,\phi) - \frac{e^2}{r}\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi), \tag{3.100}$$

where $\mathbf{L}^2$ is the orbital angular momentum operator.

3.10.1 Separation of Variables and Radial Equation

1. Show that the wavefunction can be separated as $\psi(r,\theta,\phi) = R(r)\,Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}$ is a spherical harmonic eigenfunction of $\mathbf{L}^2$ and $\mathbf{L}_z$, with:

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)\,Y_{\ell m}, \qquad \mathbf{L}_z Y_{\ell m} = \hbar m\,Y_{\ell m}. \tag{3.101}$$

Deduce that the radial Schrödinger equation for $R(r)$ is:

$$-\frac{\hbar^2}{2m_e}\left[\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER. \tag{3.102}$$

2. Let $u(r) = rR(r)$. Show that the equation becomes:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u(r) = Eu(r). \tag{3.103}$$

Define the parameter κ as:

$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}. \tag{3.104}$$

Show that introducing the dimensionless variable $\rho = \kappa r$, the equation takes the form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u(\rho), \tag{3.105}$$

where $\rho_0 = \dfrac{m_e e^2}{\hbar^2\kappa}$.

3. Propose the ansatz:

$$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho), \tag{3.106}$$

and show that $v(\rho)$ satisfies the differential equation[10]:

$$\rho\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2} + (2\ell+2-\rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0. \tag{3.107}$$

4. Expanding $v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k$, show that the series generally diverges at infinity unless it terminates at a finite order. Deduce that the termination condition is:

$$\rho_0 = 2n, \qquad \text{where } n = \hat{k} + \ell + 1 \in \mathbb{N}^*. \tag{3.108}$$

[10] This is a confluent hypergeometric equation.

5. Derive the expression for the bound energy levels of the hydrogen atom:

$$\kappa_n = \frac{m_e e^2}{\hbar^2} \cdot \frac{1}{2n} \quad \Rightarrow \quad E_n = -\frac{\hbar^2 \kappa_n^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2} \cdot \frac{1}{n^2}. \tag{3.109}$$

6. What is the degeneracy of each energy level E_n? Show that it is n^2 by considering the possible values of ℓ (from 0 to $n-1$) and m (from $-\ell$ to $+\ell$). Explain why, in this non-relativistic model, the energy depends only on n and not on ℓ.

3.10.2 Ground State ($n = 1$) and Radial Properties

7. For the ground state ($n = 1$, $\ell = 0$), show that the normalized radial wavefunction is:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}}\, e^{-r/a_0}. \tag{3.110}$$

Deduce the full expression for $\psi_{1,0,0}(r, \theta, \phi)$ and verify its normalization $\int |\psi_{1,0,0}|^2 \mathrm{d}^3x = 1$ (note that $Y_0^0 = 1/\sqrt{4\pi}$).

8. Calculate the radial probability density $P(r) = 4\pi|R_{1,0}(r)|^2 r^2$ and sketch its qualitative profile as a function of r. Interpret the physical meaning of this density (most probable location of the electron).

9. Show that the expectation value of the distance $\langle r\rangle$ between the electron and the nucleus, as well as the variance $(\Delta r)^2$, are given by:

$$\langle r\rangle = \frac{3}{2}a_0, \qquad (\Delta r)^2 = \langle r^2\rangle - \langle r\rangle^2 = \frac{3}{2}a_0^2 - \left(\frac{3}{2}a_0\right)^2. \tag{3.111}$$

(Hint: Use the integral $\int_0^\infty r^n e^{-2r/a_0} dr = n!(a_0/2)^{\,n+1}$ and verify the results.)

10. (Optional) Introduce the momentum representation. Compute the Fourier transform $\tilde{\psi}_{1,0,0}(\mathbf{p})$ of the ground state and interpret the associated momentum distribution (square modulus). What are the expectation values of the momentum $\langle \mathbf{p}\rangle$ and its square $\langle p^2\rangle$?

11. *Interpretation*: Briefly discuss how the $1/n^2$ dependence of the energy levels E_n explains the fine structure of hydrogen spectral lines and the concept of the principal quantum number.

3.11 Toward a Relativistic Formalism (QM, SR) ★★★★★

(Solution) **Conventions and notations.** Natural units $c = \hbar = 1$. Metric $g_{\mu\nu} = (-,+,+,+)$. Einstein summation convention on repeated indices. $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$, $\Box = \partial_\mu \partial^\mu$, $\not\partial \equiv \gamma^\mu \partial_\mu$. Hermitian adjoint $\dagger$. Dirac adjoint $\bar{\psi} = \psi^\dagger \gamma^0$. Spatial vectors are in bold. Fine-structure constant $\alpha = e^2/(4\pi)$. We assume that the student is familiar with the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ and the orbital angular momentum operators $\mathbf{L}$.

3.11.1 From Klein–Gordon to Dirac (algebraic factorization, historical method)

1. **Scalar equation.**

   (a) Start from the relativistic relation $E^2 = \mathbf{p}^2 + m^2$. Apply the quantum prescriptions $E \mapsto i\partial_t,\ \mathbf{p} \mapsto -i\nabla$ and show that any complex scalar field $\phi(x)$ satisfies

   $$(\Box + m^2)\,\phi = 0.$$

   (b) After your derivation, explain in 2–4 lines why the fact that the equation is second order in ∂_t makes a simple probabilistic interpretation delicate (sign of the temporal density of current not guaranteed, necessity to interpret ϕ as a field).

2. **Dirac's idea: factorization.**

   (a) Dirac seeks a linear operator D (with constant coefficients) such that, for some D',

   $$D\,D' = \Box + m^2.$$

   Justify that one can look for D of the form

   $$D = iA^\mu \partial_\mu + B,$$

   with A^μ, B independent of x. Argue why A^μ, B cannot be scalars and must act on a higher-dimensional space (intuitively: anticommutation necessary to obtain $\partial_\mu \partial^\mu$).

   (b) Impose the composition $DD' = \Box + m^2$ and, by regrouping the terms in second derivatives, first derivatives, and terms without derivatives, show that one obtains the algebraic constraints

   $$\{A^\mu, A^\nu\} = 2g^{\mu\nu}\,\mathbb{I}, \qquad A^\mu B + BA^\mu = 0, \qquad B^2 = m^2\mathbb{I},$$

   where $\{\cdot,\cdot\}$ denotes the anticommutator and $\mathbb{I}$ the identity on the space on which A^μ, B act.

3. **Guided interpretation.**

   (a) Explain in a few lines the role of anticommutation in recovering the scalar operator $\Box$ (i.e. cancellation of the cross terms).

   (b) Explain why the appearance of non-commutative objects naturally introduces additional degrees of freedom (spin, and later particle/antiparticle) — try to formulate physically what these components represent.

3.11.2 Construction of Dirac matrices, Clifford algebra and Lagrangian formalism

1. Set $B = m\beta$, $A^0 = \beta$, $A^i = \beta\alpha^i$ and define $\gamma^0 = \beta, \gamma^i = \beta\alpha^i$. Show that the linear equation
$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$
produces (via composition with $(i\gamma^\mu\partial_\mu + m)$) the operator $(\Box + m^2)\mathbb{I}$ if and only if
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{I}.$$

2. Give a brief definition (in your own words) of the Clifford algebra $\mathrm{Cl}(1,3)$ and compute $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Indicate a useful property of $\sigma^{\mu\nu}$ (infinitesimal generators of Lorentz transformations on spinors).

3. **Minimal dimension.** Argue succinctly why the smallest faithful representation of $\mathrm{Cl}(1,3)$ over $\mathbb{C}$ has dimension 4; consequently ψ is a four-component spinor. If the representation argument is unknown, give a concrete example (matrices 4×4) showing that 2×2 matrices cannot satisfy all relations.

4. **Dirac representation.** One gives
$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$
   (a) Verify the anticommutation relations in this basis.
   (b) Write $\beta,\ \alpha^i$ in 2×2 blocks and explain the decomposition $\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix}$; specify which are the "large" and "small" components for a particle at rest.

5. **Dirac Lagrangian.** Consider
$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$$
   (a) Vary $\mathcal{L}_D$ with respect to $\bar{\psi}$ and ψ (treated as independent variables) and recover the Dirac equation and its adjoint.
   (b) Show the global invariance $\psi \mapsto e^{i\alpha}\psi$ and, by Noether's theorem, compute the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$.

3.11.3 Plane waves, projectors, Gordon identity and positivity of density

1. Look for solutions $\psi(x) = u(p)e^{-ip\cdot x}$ and $\psi(x) = v(p)e^{+ip\cdot x}$. Show
$$(\not{p} - m)u(p) = 0, \qquad (\not{p} + m)v(p) = 0.$$

2. For $\mathbf{p} = \mathbf{0}$, solve explicitly and exhibit the two spin states for $E = +m$ and the two for $E = -m$. Interpret these solutions in terms of particle/antiparticle.

3. Define the projectors
$$\Lambda_\pm(p) = \frac{\pm\not{p} + m}{2m},$$
show $\Lambda_\pm^2 = \Lambda_\pm$, $\Lambda_+\Lambda_- = 0$, and establish the resolutions of the identity
$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m,$$
for the normalization $u_s^\dagger u_{s'} = 2E\delta_{ss'}$.

4. Show the Gordon identity

$$\bar{u}'(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}'(p')\big((p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu\big)u(p),$$

and briefly explain why the term in $\sigma^{\mu\nu}$ is associated with the magnetic moment (factor $g = 2$ at tree level).

5. In the Dirac representation, show that

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \|\varphi\|^2 + \|\chi\|^2 \geq 0,$$

and comment why this allows a probabilistic interpretation of ψ (contrary to the scalar case).

3.11.4 Minimal coupling and non-relativistic limit (step-by-step procedure)

1. Promote the global $U(1)$ symmetry to a local symmetry and introduce the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$. Write the Dirac–Maxwell Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

show that the interaction term is written $-q\bar{\psi}\gamma^\mu\psi A_\mu$ and identify the current $j^\mu = q\bar{\psi}\gamma^\mu\psi$.

2. **Non-relativistic limit (constructive procedure).**

   (a) Write $\psi(\mathbf{x},t) = e^{-imt}\begin{pmatrix}\varphi(\mathbf{x},t)\\ \chi(\mathbf{x},t)\end{pmatrix}$ and insert into the Dirac equation coupled to $A^\mu = (V, \mathbf{A})$. Write explicitly the exact system for φ, χ.

   (b) Under the non-relativistic assumption ($E \simeq m + \mathcal{E},\ \mathcal{E} \ll m$) and $\|\chi\| \ll \|\varphi\|$, isolate χ to order $1/m$ (show the iteration) and reinject into the equation for φ.

   (c) Expand up to order $1/m^2$ to obtain the Pauli effective Hamiltonian

$$H_{\text{eff}} = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + qV - \frac{q}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} + H_{\text{Darwin}} + H_{\text{SO}} + \mathcal{O}(1/m^3),$$

   and show explicitly (step by step) where the Darwin and spin-orbit terms come from.

   (d) Physically interpret each term (magnetic moment $g = 2$, zitterbewegung $\rightsquigarrow$ Darwin, relativistic kinetic correction).

3.11.5 Angular operators and construction of spherical spinors $\Omega_{\kappa m}$ — explicit vectorial notations

Notation conventions (emphasized). We will systematically denote operator vectors by a bold letter with an arrow: $\vec{\mathbf{L}}$, $\vec{\mathbf{S}}$, $\vec{\mathbf{J}}$, $\vec{\mathbf{\Sigma}}$. The arrow reminds that we are dealing with the three real components acting on $\mathbb{R}^3$. The Hilbert space is

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4,$$

i.e. spatial functions of $\mathbf{r}$ valued in $\mathbb{C}^4$ (Dirac spinors). The operators are distributed as follows:

- $\vec{\mathbf{L}}$ acts on the $L^2(\mathbb{R}^3)$ part (angular dependence);

- $\vec{\mathbf{\Sigma}}$ acts on the $\mathbb{C}^4$ part (spinorial component index);
- $\vec{\mathbf{S}} = \frac{1}{2}\vec{\mathbf{\Sigma}}$ is the effective spin operator.

1. **Orbital angular momentum $\vec{\mathbf{L}}$.**
   (a) Definition (differential operator on the spatial part)
   $$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}, \qquad \vec{\mathbf{p}} = -i\nabla.$$
   (b) Fundamental relations:
   $$[\mathbf{L}_i, \mathbf{L}_j] = i\,\varepsilon_{ijk}\mathbf{L}_k, \qquad [\vec{\mathbf{L}}^2, \mathbf{L}_i] = 0.$$
   (These relations are the same as those used for the spherical harmonics $Y_{\ell m}$.)
2. **Spin and operator $\vec{\mathbf{\Sigma}}$.**
   (a) Matrix definition (operator acting on $\mathbb{C}^4$):
   $$\vec{\mathbf{\Sigma}} = \begin{pmatrix} \vec{\boldsymbol{\sigma}} & 0 \\ 0 & \vec{\boldsymbol{\sigma}} \end{pmatrix}, \qquad \text{i.e.} \qquad \mathbf{\Sigma}_i = \begin{pmatrix} \boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i \end{pmatrix},$$
   where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli 2×2 matrices.
   (b) The spin operator is
   $$\vec{\mathbf{S}} \equiv \tfrac{1}{2}\,\vec{\mathbf{\Sigma}} \quad (\text{acts on } \mathbb{C}^4).$$
   (c) Algebraic properties:
   $$[\mathbf{S}_i, \mathbf{S}_j] = i\,\varepsilon_{ijk}\mathbf{S}_k, \qquad \vec{\mathbf{S}}^2 = \tfrac{3}{4}\,\mathbb{I}_4.$$
   (d) *Space remark:* $\vec{\mathbf{\Sigma}}$ does not act on the variable $\mathbf{r}$ (it does not differentiate), it acts only on the spinorial component index. Thus $\vec{\mathbf{S}}$ and $\vec{\mathbf{L}}$ operate on distinct factors of $\mathcal{H}$.
3. **Total angular momentum $\vec{\mathbf{J}}$.**
   $$\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}.$$
   We have
   $$\vec{\mathbf{J}}^2 = \vec{\mathbf{L}}^2 + \vec{\mathbf{S}}^2 + 2\,\vec{\mathbf{L}} \cdot \vec{\mathbf{S}},$$
   and for a central Hamiltonian $H = H(r)$ (potential depending only on r), $\vec{\mathbf{J}}^2$ commutes with H. The eigenvalues of $\vec{\mathbf{J}}^2$ are $j(j+1)$ with $j = \ell \pm \frac{1}{2}$.
4. **Auxiliary operator $\hat{K}$ and its usefulness.**
   $$\hat{K} \equiv \beta\big(\vec{\mathbf{\Sigma}} \cdot \vec{\mathbf{L}} + 1\big).$$
   - $\hat{K}$ is Hermitian and commutes with the Dirac Hamiltonian in a central potential (reason: $\hat{K}$ is built from angular operators and β, and the commutators with $\boldsymbol{\alpha} \cdot \vec{\mathbf{p}}$ and $V(r)$ vanish — see detailed proof in the correction).
   - The eigenvalues are denoted $\kappa = \pm\big(j + \frac{1}{2}\big)$. κ is therefore a good quantum number to classify states.
5. **Construction of spherical spinors $\Omega_{\kappa m}(\theta, \varphi)$.**

(a) Logic: as for the construction of $Y_{\ell m}$ (eigenfunctions of $\vec{\mathbf{L}}^2$ and L_z), we construct angular functions that are eigenfunctions of $\vec{\mathbf{J}}^2,\ J_z$ by coupling $\ell \otimes s$ (here $s = \frac{1}{2}$).

(b) Constructive formula (coupling via Clebsch–Gordan coefficients):

$$\boxed{\Omega_{\kappa m}(\theta,\varphi) \;=\; \sum_{m_\ell, m_s} \langle \ell m_\ell; \tfrac{1}{2} m_s \mid jm\rangle \, Y_{\ell m_\ell}(\theta,\varphi)\, \chi_{m_s}},$$

where $\chi_{+\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$ are the eigenvectors of spin $1/2$ (acting in $\mathbb{C}^2$). In practice, $\Omega_{\kappa m}$ is a two-component spinor (we then place these two components in the upper or lower block of a four-component Dirac spinor, depending on the convention).

(c) Correspondence rule $(\ell, j) \leftrightarrow \kappa$ (standard convention):

$$\begin{cases} \kappa = -(\ell+1) & \text{if } j = \ell + \frac{1}{2}, \\ \kappa = +\ell & \text{if } j = \ell - \frac{1}{2}. \end{cases}$$

This rule allows one to associate a unique κ to each pair (ℓ, j).

(d) Useful properties:

$$\text{(orthonormality)} \quad \int \Omega^\dagger_{\kappa m}(\theta,\varphi)\,\Omega_{\kappa' m'}(\theta,\varphi)\,\mathrm{d}\Omega = \delta_{\kappa\kappa'}\delta_{mm'},$$

$$\text{(radial action)} \quad (\vec{\sigma}\cdot\hat{r})\,\Omega_{\kappa m}(\theta,\varphi) = -\,\Omega_{-\kappa m}(\theta,\varphi).$$

The second identity is fundamental: it explains why, in the radial separation, the angular component of the large component (associated with κ) is linked to the angular component of the small component (associated with $-\kappa$).

6. **Radial separation — why the $1/r$ factor.**

$$\psi_{E,\kappa,m}(\mathbf{r}) = \frac{1}{r}\begin{pmatrix} F_{E\kappa}(r)\,\Omega_{\kappa m}(\theta,\varphi) \\ i\,G_{E\kappa}(r)\,\Omega_{-\kappa m}(\theta,\varphi)\end{pmatrix}.$$

- The factor $1/r$ is chosen for the same reason as in the Schrödinger problem: it simplifies the radial operator (avoids the appearance of terms in $2/r$ coming from the radial derivative) and makes it possible to obtain a radial system for F, G without additional coupling terms complicating the power series.
- The factor i in front of G is a convention that makes the radial equations real in most representations.

7. **Short exercises (to practice).**

(a) Construct explicitly $\Omega_{\kappa m}$ for $\kappa = 1$ (state $s_{1/2}$, $\ell = 0$) and check orthonormality.

(b) Write the combination for $\kappa = -1$ ($p_{1/2}, \ell = 1, j = 1/2$) using simple Clebsch–Gordan coefficients and verify the property $(\vec{\sigma}\cdot\hat{r})\Omega_{\kappa m} = -\Omega_{-\kappa m}$.

(c) Verify that $\hat{K}\Omega_{\kappa m} = \kappa\,\Omega_{\kappa m}$ (show the main steps).

Final remark. This explicit formulation — operators in bold $\vec{\cdot}$ with arrows, spin matrices $\vec{\mathbf{\Sigma}}$ on $\mathbb{C}^4$, and radial separation with the $1/r$ factor — makes the construction of spherical spinors $\Omega_{\kappa m}$ and their role in the separation of the Dirac equation completely natural and traceable, exactly as it is done for $\vec{\mathbf{L}}$ and the spherical harmonics $Y_{\ell m}$.

3.11.6 Relativistic hydrogen: radial separation and derivation of levels without specialized theory of special functions

Approach note (for L3). It is possible to avoid a deep theoretical development of hypergeometric functions by proceeding via a *power-series method* strictly analogous to that used for nonrelativistic hydrogen: we propose a factorization of singularities (at zero and at infinity), we write a power series for the remaining part, we obtain a recurrence relation for the coefficients, and we impose termination of the series to guarantee normalizability. This method is constructive and suitable for an L3 course: it requires algebraic manipulations but not prior knowledge of Kummer/Whittaker functions. The questions below guide this elementary calculation.

1. **Stationary equation and angular separation.**
   (a) Write the stationary equation in the Coulomb potential $V(r) = -\dfrac{Z\alpha}{r}$:
   $$\big(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m + V(r)\big)\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$
   (b) We use the spherical spinors $\Omega_{\kappa m}(\theta,\varphi)$.
   $$\psi_{E,\kappa,m}(\mathbf{r}) = \frac{1}{r}\begin{pmatrix} F_{E\kappa}(r)\,\Omega_{\kappa m}(\theta,\varphi) \\ i\,G_{E\kappa}(r)\,\Omega_{-\kappa m}(\theta,\varphi)\end{pmatrix},$$
   and justify the form (factor $1/r$, choice of indices κ, $\pm\kappa$).
2. **Radial system.** By projecting onto the angular components (guided steps), show that $F(r)$, $G(r)$ satisfy
   $$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}r} + \dfrac{\kappa}{r}F = \big(m + E - V(r)\big)G, \\ \dfrac{\mathrm{d}G}{\mathrm{d}r} - \dfrac{\kappa}{r}G = \big(m - E + V(r)\big)F. \end{cases}$$
3. **Decoupling and second-order equation (algebraic).**
   (a) Isolate G from the first equation:
   $$G = \frac{1}{m+E-V}\Big(\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{\kappa}{r}F\Big),$$
   then differentiate this expression and substitute into the second equation to obtain a second-order equation for F. Carry out the algebraic simplifications step by step (take care with signs and factors).
   (b) For $V(r) = -Z\alpha/r$, reorganize the equation by highlighting the singularities at $r = 0$ (terms in $1/r^2$, $1/r$) and the constant term at infinity.
4. **Change of variables and factorization.** For bound states ($|E| < m$) define
   $$\lambda \equiv \sqrt{m^2 - E^2}\ (> 0), \qquad \rho \equiv 2\lambda r.$$
   Show that, after the change of variable, the equation for $F(\rho)$ admits the asymptotic factorization
   $$F(\rho) = \rho^{\gamma} e^{-\rho/2}\, f(\rho),$$
   where $\gamma > 0$ is the exponent governing the behavior near $\rho = 0$ and $e^{-\rho/2}$ ensures exponential decay at infinity.

(a) Determine γ by solving the indicial equation in the neighborhood $\rho \to 0$ and show

$$\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}.$$

(b) Note and discuss the necessary physical condition $\kappa^2 > (Z\alpha)^2$ for γ to be real (remark on the "supercritical" region if $Z\alpha$ approaches 1).

5. **Series method (termination and quantization).**

   (a) Write $f(\rho) = \sum_{n=0}^{\infty} a_n \rho^n$. By inserting into the equation obtained for f, deduce the recurrence relation linking a_{n+1} and a_n. Write explicitly the general form of the recurrence (show the steps).

   (b) Analyze the behavior of the series as $\rho \to \infty$. Argue (taking inspiration from the nonrelativistic example) that the series will diverge exponentially unless it *terminates* (i.e. there exists n_r such that $a_{n_r+1} = 0$ and $a_n = 0$ for $n > n_r$). This termination condition is the quantization condition.

   (c) Show that the termination condition rewrites, after algebraic simplification, in the simple form

$$n_r + \gamma = \frac{Z\alpha E}{\lambda},$$

   with $n_r \in \mathbb{N}$ (radial quantum number).

6. **Algebraic solution for the energies.**

   (a) Set $N \equiv n_r + \gamma > 0$ and rewrite the previous relation as

$$\lambda = \frac{Z\alpha E}{N}.$$

   Square and use $\lambda^2 = m^2 - E^2$ to obtain an algebraic relation in E^2. Show that one arrives at

$$E^2\left(1 + \frac{(Z\alpha)^2}{N^2}\right) = m^2.$$

   (b) Deduce the closed expression of the bound levels

$$\boxed{E_{n_r,\kappa} = m\left[1 + \frac{(Z\alpha)^2}{\left(n_r + \sqrt{\kappa^2 - (Z\alpha)^2}\right)^2}\right]^{-1/2}}.$$

7. **Relate to usual notations and fine structure.**

   (a) Define the principal quantum number $n \equiv n_r + |\kappa|$. Show that for $\kappa = \pm(j + \frac{1}{2})$ this definition gives $n \in \mathbb{N}^*$ and justifies the usual numbering of levels.

   (b) By rearranging one obtains the classical form

$$E_{n,j} = m\left[1 + \frac{(Z\alpha)^2}{\left(n - \delta_j\right)^2}\right]^{-1/2}, \qquad \delta_j \equiv j + \tfrac{1}{2} - \sqrt{\left(j + \tfrac{1}{2}\right)^2 - (Z\alpha)^2}.$$

   Briefly present (2–4 lines) how the degeneracy in ℓ of the nonrelativistic theory is partially lifted (dependence on j only).

(c) Expand $E_{n,j}$ for $Z\alpha \ll 1$ up to order $(Z\alpha)^4$ and show that

$$E_{n,j} \simeq m - \frac{m(Z\alpha)^2}{2n^2} - \frac{m(Z\alpha)^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \mathcal{O}((Z\alpha)^6).$$

(d) (Numerical exercise) For $Z = 1$ and $n = 1, j = \frac{1}{2}$, estimate the fine-structure correction in eV (approximate value). Hint: $mc^2 \simeq 511\,\mathrm{keV}$, $\alpha \simeq 1/137$.

3.12 Pöschl–Teller Potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$ (QM) ★★★

(Solution)

We consider a quantum particle subject to an attractive potential of the form

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)}, \tag{3.112}$$

called the Pöschl–Teller potential, where $\alpha > 0$. This potential admits a finite number of bound states and allows for an exact solution of the Schrödinger equation.
The Hamiltonian of the system is given by

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}) = \frac{\mathbf{P}^2}{2m} - \frac{V_0}{\cosh^2(\alpha \mathbf{X})} \tag{3.113}$$

1. Write the time-independent Schrödinger equation for a wave function $\psi(x)$:

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{V_0}{\cosh^2(\alpha x)}\,\psi(x) = E\,\psi(x). \tag{3.114}$$

2. Show that the substitution $u = \tanh(\alpha x)$ yields:

$$\psi'(x) = \alpha\,(1-u^2)\,\frac{\mathrm{d}\phi}{\mathrm{d}u}, \quad \psi''(x) = \alpha^2\Big((1-u^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}\phi}{\mathrm{d}u}\Big)(1-u^2) \tag{3.115}$$

with $\phi(u) = \psi(x(u))$.

3. Deduce that the equation in $u \in]-1,1[$ becomes

$$(1-u^2)\,\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\,\frac{\mathrm{d}\phi}{\mathrm{d}u} + \Big[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\Big]\phi = 0, \tag{3.116}$$

and express λ, μ in terms of $V_0, \alpha, m, \hbar, E$.

4. Identify λ, μ. We seek a solution of the form $\phi(u) = (1-u^2)^{\frac{\mu}{2}}P(u) = Q(u)P(u)$.

Hint: Express $Q'(u)$ in terms of $Q(u)$ to simplify the calculations.

Show that $P :]-1,1[\to \mathbb{R}$ satisfies the differential equation

$$(1-u^2)P'' - 2(\mu+1)uP' + [\lambda(\lambda+1) - \mu(\mu+1)]P = 0 \tag{3.117}$$

5. By examining the behavior of the solution at the endpoints $u \to \pm 1$, we aim to show that $P(u)$ must be a polynomial for $\phi \in L^2([-1,1])$. We will proceed by contradiction, assuming P is not a polynomial.

   (a) Show that the normalization condition reads

$$\int_{-1}^{1} \frac{|\phi(u)|^2}{1-u^2} < \infty \tag{3.118}$$

   (b) P is analytic on $]-1,1[$. Write $P(u) = \sum_{p=0}^{\infty} a_p u^p$. Derive a recurrence relation between a_{p+2} and a_p and show that

$$a_{p+2} \underset{\infty}{\sim} a_p \tag{3.119}$$

(c) Conclude using the Riemann criterion for integrals[11].

6. It follows that $n = \deg P \in \mathbb{N}$. We therefore write

$$P(u) = \sum_{p=0}^{n} a_p u^p \tag{3.120}$$

Use the recurrence relation between a_{p+2} and a_p to show that $\mu = \lambda - n$[12].

7. Deduce the quantized energy levels E_n as

$$E_n = -\frac{\hbar^2\alpha^2}{2m}(\lambda - n)^2, \quad n = 0, 1, \ldots, \lfloor\lambda\rfloor, \tag{3.121}$$

where $\lambda(\lambda+1) = \frac{2mV_0}{\hbar^2\alpha^2}$.

8. Show that the number of bound states is finite: $N = \lfloor\lambda\rfloor + 1$.

9. Provide a physical explanation for why the number of bound states is finite, despite the "bottomless" shape of the potential. Discuss the connection with the asymptotic decay of the potential.

[11] For full mathematical rigor, one should also invoke the theorem on equivalence of divergent series.

[12] We consider only bound states, hence $\mu > 0$.

3.13 Larmor Power and Electrodynamic Instability of the Classical Atom [5] (EM) ★★

(Solution) A confined (and thus accelerated) charge emits electromagnetic radiation. We now examine in more detail some consequences of classical Electromagnetic laws combined with those of dynamics (hence: *Electrodynamics*), and in particular, show that the classical atom is fundamentally unstable: the electron localized in the atom emits radiation and, as a result, gradually loses energy.

The description below relies on the assumption that the radiation effect is a minor phenomenon, although it ultimately leads to dramatic conclusions. We will therefore start with an ordinary dynamical description, to which we will add the perturbative effects of the source's (the confined electron's) radiation on its own motion.

3.13.1 Calculation of the radiation reaction force $\mathbf{F}_{\text{rad}}$.

The Larmor power is the power lost by an accelerated charge. We will deduce a radiation reaction force $\mathbf{F}_{\text{rad}}$ from it, which leads to dramatic consequences.

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{2q^2 a^2}{3c^3} \tag{3.122}$$

1. Write the work $\mathrm{d}E_{\text{at}} = \mathrm{d}W$, equal to the change in atomic energy over a time $\mathrm{d}t$ due to the radiation force $\mathbf{F}_{\text{rad}}$.

2. Write the energy variation of the atom over a time $\mathrm{d}t$ due to the radiated power of the electron.

3. By integrating by parts and assuming periodic motion, show that[13],

$$\mathbf{F}_{\text{rad}} = \frac{2\vartheta^2}{3c^3}\ddot{\mathbf{v}} \tag{3.123}$$

4. Apply Newton's second law with the previously calculated $\mathbf{F}_{\text{rad}}$[14] and a restoring force $\mathbf{F} = -m\omega_0^2\mathbf{r}$. Seeking a solution of the form $\mathbf{r}(t) = \mathrm{Re}\{\mathbf{r}_0 e^{i\omega t}\}$, and letting

$$\omega = \omega_0(1 + \alpha(\omega_0\tau) + o(\omega_0\tau)), \alpha \in \mathbb{R} \tag{3.124}$$

show that the solution is a damped oscillator.
N.B. Given $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s, $\omega_0 = 3 \times 10^{15}$ rad.s^{-1}. Comment.

3.13.2 Conceptual issues raised by the radiation reaction force $\mathbf{F}_{\text{rad}}$.

The radiation reaction force $\mathbf{F}_{\text{rad}}$ written above is conceptually pathological, as the following analysis shows. Using the notations of Section 1.5, Volume I, the Abraham-Lorentz equation for a particle of charge e and mass m subjected to a force $\mathbf{F}$ (with $\mathbf{v} = \dot{\mathbf{r}}$) is:

$$m\ddot{\mathbf{r}} = m\tau\,\dddot{\mathbf{r}} + \mathbf{F}, \tag{3.125}$$

[13] Where $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$

[14] Note the appearance of a force depending on the derivative of the acceleration. We will study in the next part the issues caused by this force.

where $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s is a characteristic time. One oddity of this equation is the appearance of a third derivative of the particle's position (defined by the radius vector $\mathbf{r}$), which is meant to represent the radiation damping effect.

Moreover, the perturbation of motion caused by this effect is fundamentally *singular*, in the sense that it alters the order of the motion equation, which changes from second to third order as soon as the charge is nonzero. In fact, it is precisely because the small parameter τ multiplies the highest derivative that the perturbation is called *singular*, by definition[15].

With these warnings in mind, we now examine the consequences of equation (3.125) as it stands, to highlight the deep conceptual issues it poses.

1. Using the standard method for solving a differential equation like (3.125), write the general expression for the acceleration $\ddot{\mathbf{r}}(t)$, assuming the acceleration at some instant t_0, $\ddot{\mathbf{r}}(t_0)$, is known.
2. Examine the particular case $\mathbf{F} = 0$, and show that the solution is physically aberrant.
3. Returning to the general solution obtained in 1 for $\mathbf{F} \neq 0$, show that the divergent solutions can formally be eliminated by a suitable choice of t_0. Comment on this choice — which, from a technical standpoint, expresses a boundary condition rather than an initial condition.
4. Deduce the regularized expression of the solution obtained in 1. Stepping back, analyze the integral kernel in this expression and verify that, in the limit of zero charge, the motion equation reduces to the standard dynamical equation.
5. To clearly exhibit the violation of a major physical principle, make a simple change of integration variable to obtain:

$$\dot{\mathbf{v}}(t) = \frac{1}{m} \int_0^{+\infty} e^{-s} \times \mathbf{F}(t + \tau s)\, \mathrm{d}s. \tag{3.126}$$

   Comment on this equation and show that it violates a physical principle.
6. To highlight this violation even more spectacularly, treat the case of a particle with zero velocity at $t = -\infty$ and subjected to a step force:

$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0, \\ \mathbf{F}_0 & \text{if } t > 0. \end{cases} \tag{3.127}$$

   Summarize these results by plotting the time evolution of the acceleration and velocity. Note that the particle starts moving... **before the force is applied**!

[15] The same phenomenon occurs in the Schrödinger eigenvalue equation, where it is Planck's constant that multiplies the highest derivative. A specific perturbation technique is used for such problems, known as the WKB (or BKW) method in the quantum context.

3.14 Geodesics in a Dispersive Optical Medium (WO, AM, EM, TH) ★★★★★

(Correction)
The goal of this exercise is to understand how light propagates in media from the variational principle, taking into account dispersion (dependence of the refractive index on the wavelength λ). This will then allow us to explain various common optical phenomena.

3.14.1 Fermat's Principle and Optical Metric

1. Derive Fermat's principle from the principle of least action. This principle can be interpreted as a geodesic for an effective metric given by:

$$\mathrm{d}s^2 = n(\mathbf{r}, \lambda)^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{3.128}$$

   which is a metric conformal to the Euclidean one: $g_{ij}(\mathbf{r}, \lambda) = n(\mathbf{r}, \lambda)^2 \delta_{ij}$ in Einstein notation.

2. Justify why this metric is suitable for light propagation in an inhomogeneous and dispersive medium.

3. Show that the optical trajectories are the geodesics of this metric.

3.14.2 Calculation of a Refractive Index $n(\mathbf{r}, \lambda)$

In any medium, the refractive index depends microscopically on the density through the electric susceptibility. Here, we want to justify, from a realistic electromagnetic model[16], that the index can be expressed in the form, with $\zeta = \frac{\omega}{\omega_0}$:

$$\boxed{n(\mathbf{r}, \zeta) \underset{\zeta \to 0}{=} 1 + \frac{1}{2} \frac{N(\mathbf{r}) e^2}{m \omega_0^2 \varepsilon_0} \left(1 + \zeta^2\right) + o(\zeta^2)}. \tag{3.129}$$

1. Recall that in an isotropic linear medium with negligible magnetic field, the refractive index satisfies $n^2 = \varepsilon_r = 1 + \chi$.

2. From the Lorentz model for a bound electron subjected to an electric field, with a single resonance ω_0, express the susceptibility $\chi(\mathbf{r}, \omega)$ as a function of the local density $N(\mathbf{r})$. In particular, show that for negligible damping γ,

$$\boxed{n^2(\mathbf{r}, \zeta) = 1 + \frac{N(\mathbf{r}) e^2}{m \omega_0^2 \varepsilon_0} \frac{1}{1 - \zeta^2}}. \tag{3.130}$$

3. Assuming $\omega \ll \omega_0$, deduce the Taylor expansion of equation (3.129).

This model will then allow the introduction of an optical metric to study the geodesics of light in the drop.

[16] The validity of the expansion depends on the medium considered. In the case of a gas or a liquid, it is usually sufficient to consider a single main resonance located in the ultraviolet. This allows a very precise approximation: the error on the index is typically less than 0.01 % in air, and about 0.1 % in water, within the visible range.

3.14.3 Calculation of $N(\mathbf{r})$ for a Gas and a Liquid

We seek to calculate $N(\mathbf{r})$ for air and water, treating each case independently. Recall that,

$$N(\mathbf{r}) = \frac{\rho(\mathbf{r})}{M}\mathcal{N}_A, \tag{3.131}$$

where ρ is the mass density, M the molar mass, and $\mathcal{N}_A$ Avogadro's number.

1. Assume air is a diatomic ideal gas. It is subject to the gravitational field $\mathbf{g} = -g\mathbf{e}_z$. We suppose hydrostatic equilibrium and that $\delta Q = 0$, i.e., the atmosphere is adiabatic.
   (a) Using the first law of thermodynamics, prove that
   $$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{g}{C_p}, \tag{3.132}$$
   and deduce $T(z)$.
   (b) Determine a differential equation for p and prove that
   $$p(z) = p_0 \left(\frac{T(z)}{T_0}\right)^{\frac{gM}{R\Gamma}}, \quad \Gamma = \frac{g}{C_p}, \tag{3.133}$$
   (c) Deduce $N(\mathbf{r}) = N(z)$ in this case.
2. In the case of a liquid, one can generally assume $\rho(\mathbf{r})$ is constant. Deduce $N(\mathbf{r})$.

3.14.4 Optical geodesics in a spherical medium

We consider an isotropic medium whose refractive index n depends only on the radial position $r = \|\mathbf{r}\|$ and the reduced frequency $\zeta = \omega/\omega_0$, via the function $n(r, \zeta)$. This framework models, for example, spherically symmetric electron density profiles $N(r)$, relevant for idealized planetary atmospheres.

In this context, light propagation is described by an optical metric, defining an infinitesimal scalar product between two vectors $\mathrm{d}x^i = (\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$:

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.134}$$

where g_{ij} is the spherical optical metric. The light trajectory is parametrized by an affine parameter s, with functions $r(s), \theta(s), \varphi(s)$.

The associated Lagrangian reads:

$$\mathcal{L} = n^2(r,\zeta)\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right), \tag{3.135}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$.

We restrict the study to the equatorial plane $\theta = \frac{\pi}{2}$, simplifying the Lagrangian to:

$$\mathcal{L} = n^2(r,\zeta)\left(\dot{r}^2 + r^2\dot{\varphi}^2\right), \tag{3.136}$$

with $\dot{\theta} = 0$.

We use for the refractive index the electromagnetic dispersion law **??**.

1. **Radial dependence of the index**

   (a) Justify that, for a fixed frequency ζ, the radial dependence of n^2 is entirely governed by the profile $N(r)$. Verify that the index is real as long as $\zeta < 1$.

   (b) Calculate the radial derivative $\partial_r n^2(r,\zeta)$, and express it in the form:

   $$\partial_r n^2(r,\zeta) = C(\zeta) \cdot N'(r), \tag{3.137}$$

   with $C(\zeta)$ an explicit constant.

   (c) Discuss the sign of $\partial_r n^2$ if $N(r)$ is a decreasing function (e.g. $N(r) = \frac{N_0}{1+\lambda r^2}$).

2. **Christoffel symbols**

   In the equatorial plane $\theta = \pi/2$, the optical metric takes the diagonal form:

   $$g_{ij} = n^2(r,\zeta) \cdot \mathrm{diag}(1, r^2). \tag{3.138}$$

   Recall the formula for the Christoffel symbols:

   $$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right). \tag{3.139}$$

   (a) Deduce the explicit expressions for $\Gamma^r_{rr}, \Gamma^r_{\varphi\varphi}, \Gamma^\varphi_{r\varphi}$ in terms of $n(r,\zeta)$, $n'(r)$, and r, with $n'(r) = \frac{\partial n}{\partial r}$.

   (b) Show that:

   $$\Gamma^r_{rr} = \frac{\partial_r n^2}{2n^2}, \qquad \Gamma^r_{\varphi\varphi} = -rn^2 + \frac{r^2 \partial_r n^2}{2}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} + \frac{\partial_r n^2}{2n^2}. \tag{3.140}$$

3. **Geodesic equations**

   Recall the geodesic equation in a Levi-Civita connection space:

   $$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s} = 0. \tag{3.141}$$

   (a) Write explicitly the geodesic equations in the equatorial plane for $r(s)$ and $\varphi(s)$, using the Christoffel symbols obtained above.

   (b) Discuss qualitatively the effect of the gradient $\partial_r n^2$ on the curvature of the trajectory (is light bent towards higher or lower index zones?).

4. **Lagrangian formalism**

   Consider the optical Lagrangian:

   $$\mathcal{L} = n^2(r,\zeta) \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right). \tag{3.142}$$

   (a) Write the Euler–Lagrange equations for the two generalized coordinates r and φ.

   (b) Show that the Euler–Lagrange equation for φ leads to a conservation law (conservation of the effective angular momentum).

5. **Consistency check**

   Verify that the equations obtained from the Lagrangian formalism coincide with those derived from geometry (Christoffel symbols).

6. **Conservation and symmetries**

(a) Show that the rotational symmetry in φ implies the conservation of optical angular momentum:

$$\ell = n^2(r,\zeta) \cdot r^2\dot{\varphi} = \text{constant}. \tag{3.143}$$

(b) Deduce a new expression for the Lagrangian in the form:

$$\mathcal{L} = n^2(r,\zeta)\dot{r}^2 + \frac{\ell^2}{n^2(r,\zeta)r^2}. \tag{3.144}$$

7. **Parameterization by angle φ**

(a) Propose a change of parameter to describe the trajectory as $r = r(\varphi)$ and show how to express $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}s}$ in terms of $\frac{\mathrm{d}r}{\mathrm{d}\varphi}$.

(b) Using the conservation of effective angular momentum $\ell = n^2(r,\zeta)r^2\dot{\varphi}$, show that:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \left(\frac{n^2(r,\zeta)r^2}{\ell}\right)^2 \left(\frac{\mathcal{L}}{n^2(r,\zeta)} - \frac{\ell^2}{n^4(r,\zeta)r^2}\right). \tag{3.145}$$

(c) Demonstrate that the Lagrangian $\mathcal{L}$ is conserved along the trajectory, i.e. $\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}s} = 0$.

(d) Physically interpret the conservation of $\mathcal{L}$ in the context of light propagation in a variable-index medium.

(e) Deduce an explicit differential equation for $r(\varphi)$ in a given medium $n(r,\zeta)$, and briefly discuss analytical or numerical solutions.

8. **Local Snell's law**

Using the conservation of effective angular momentum, show that an analogue of Snell's law holds locally:

$$n(r,\zeta) \cdot r\sin\alpha = \text{constant}, \tag{3.146}$$

where α is the angle between the trajectory tangent and the radial direction.

9. **Numerical method**

(a) Propose a numerical scheme to integrate the differential equation for $r(\varphi)$, with initial conditions $r(0) = r_0$, $\frac{\mathrm{d}r}{\mathrm{d}\varphi}(0) = v_0$.

(b) Discuss the impact of the reduced frequency ζ on the trajectory (angular dispersion).

3.14.5 First application: the rainbow as a geometric manifestation of dispersion

We model a water droplet as a homogeneous sphere of radius R and refractive index $n(\omega)$ computed in section 3.14.2, immersed in air with index $\simeq 1$ (see Fig. 3.4[17]).

1. **Refractive index of water.**

Recall the expression of $n(\mathbf{r},\omega) = n(\omega)$ for a liquid.

2. **Geometric modelling of a primary rainbow.**

[17] Source.

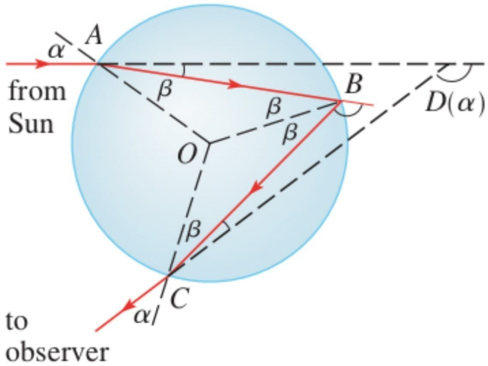

Figure 3.4: Geometric diagram of the incidence, refraction, and deviation angles for a light ray crossing a spherical water droplet, introducing the total deviation angle $\Theta - \pi = D$.

(a) Consider an incident ray from the Sun, entering the droplet with incidence angle α, refracted according to Snell's law, reflected once inside, then refracted again upon exit. Justify that the ray remains straight inside the droplet (constant index), and that deflections occur only at spherical surfaces.

(b) Let $\Theta(\omega, \alpha)$ be the total deviation angle. Show that:

$$\Theta(\omega, \alpha) = 2\alpha - 4 \arcsin\left(\frac{\sin\alpha}{n(\omega)}\right) + \pi. \tag{3.147}$$

(c) Show that $\Theta(\omega, \alpha)$ has a minimum for some critical angle $\alpha_c(\omega)$. Prove that:

$$\alpha_c(\omega) = \arcsin\left(\sqrt{\frac{4 - n^2(\omega)}{3}}\right). \tag{3.148}$$

Deduce that rays concentrate around a particular exit angle $\Theta_{\text{min}}(\omega)$, resulting in a maximum observed light intensity in this direction (see Fig. 3.5).

(d) **Numerical application.** For $n \simeq 1.33$, calculate the angle D_{min}.

(e) Study the function $\Theta_{\text{min}}(\omega)$ and explain the rainbow phenomenon.

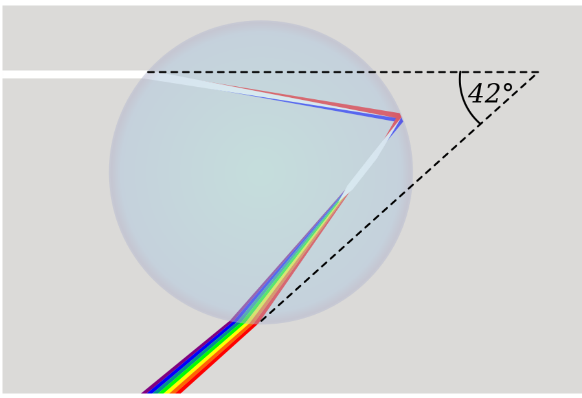

Figure 3.5: Formation of the rainbow and the angle $\Theta_{\text{min}}(\omega)$ (Wikipedia).

3.14.6 Second application: colours of soap bubbles (thin film interference) [1]

Here we study the iridescent colours seen on soap bubbles, explained by interference between waves reflected within a very thin liquid layer modelled as a thin film.

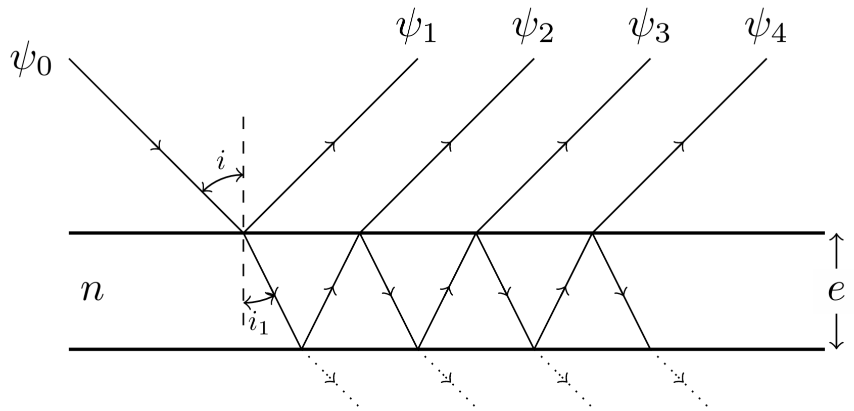

Figure 3.6: Thin film model of a soap bubble.

The bubble is represented by a liquid layer of index $n \simeq \frac{4}{3}$ and thickness e, enclosed between two air/liquid/air interfaces (see Fig. 3.6)[18]. The air index is taken as 1. The incident light is monochromatic of wavelength λ and nearly normal incidence[19].

1. **Multiple reflections and interference.**

   (a) Explain why light reflected by the bubble decomposes into an infinite sum of waves successively reflected and transmitted at the two interfaces.

   (b) Show that the phase difference between two consecutive reflected waves is:

$$\varphi = \frac{4\pi n e}{\lambda}. \tag{3.149}$$

2. **Fresnel coefficients and reflected amplitude.**

   Let $r = \frac{1-n}{1+n}$ and $t = \frac{2}{1+n}$ be the reflection and transmission coefficients at the air-soap interface (normal incidence).

   (a) Calculate numerically r and t for $n = \frac{4}{3}$.

   (b) Write the total reflected amplitude ψ_r as the sum of the geometric series of multiple waves, clearly showing each reflection and transmission contribution. Show that:

$$\psi_r = \psi_0 \left[r + \frac{t^2 r e^{i\varphi}}{1 - r^2 e^{i2\varphi}} \right]. \tag{3.150}$$

3. **Reflected intensity and interference conditions.**

   (a) Deduce the expression of the reflected intensity $I_r = \frac{|\psi_r|^2}{2}$.

[18]In the diagram, i is given but we can take $i = 0$.

[19]This problem illustrates how thin film interference, modulated by microscopic variations of thickness and refractive index, can generate spectacular visual effects similar to rainbows.

(b) Show that this intensity exhibits maxima and minima as a function of φ, and determine the conditions on λ for destructive and constructive interference.

4. **Numerical application: thickness $e = 0.3\ \mu m$.**

(a) Determine the values of λ in the interval $[0.4, 0.8]\ \mu m$ for which reflection is minimal.

(b) What is the dominant colour perceived by the human eye in this case?

(c) Qualitatively study what happens when $e = 0.03\ \mu m$ and when $e = 30\ \mu m$. Interpret the optical consequences of these two situations.

5. **Extension: position- and frequency-dependent index, and white light illumination.**

We now consider that the optical index in the bubble depends both on the radial position $R \in [0, e]$ (thickness direction) and the frequency ω via the reduced variable $\zeta = \frac{\omega}{\omega_0}$, with ω_0 a characteristic frequency.

The local oscillator density varies radially as:

$$N(R) = N_0 \left(1 - \mu R^2\right), \tag{3.151}$$

with $\mu > 0$ a spatial variation parameter.

(a) Recall the expanded form of the index for $\zeta \to 0$ (eq. 3.129) and explain how the radial variation $N(R)$ locally affects the optical index.

(b) Taking into account the index variation along R, show that:

$$\boxed{\varphi(\omega) = \frac{2\omega}{c} \int_0^e n\left(R, \frac{\omega}{\omega_0}\right) \mathrm{d}R.} \tag{3.152}$$

(c) Explain why illumination with white light (broad ω spectrum) can generate complex colourful patterns (rainbow-like) on the bubble surface.

(d) For $N(R) = N_0(1 - \mu R^2)$, compute explicitly the integral contribution to the phase shift $\varphi(\omega)$.

(e) Show that the extremisation condition $\frac{\mathrm{d}I}{\mathrm{d}\varphi} = 0$ implies:

$$a\omega^3 - b\omega + p\pi = 0, \quad p \in \mathbb{Z}, \tag{3.153}$$

where a, b are expressed in terms of $c, e, N_0, m, \omega_0, \varepsilon_0$.

(f) Assuming $a > b$[20], study:

$$f_p(\omega) = a\omega^3 - b\omega + p\pi, \quad p \in \mathbb{Z}. \tag{3.154}$$

Deduce that there exists a unique $\omega_p \in \mathbb{R}$ satisfying $f_p(\omega_p) = 0$.

(g) Study the sequence (ω_p).

(h) Summarise what has been achieved here.

3.14.7 Third application: mirages [2]

Fermat's principle states that light travels between two points along a path that minimises travel time. The air's refractive index near the ground depends only on altitude, $n(z)$. We consider a light ray starting from height h, with an initial downward angle θ. Points in the xOz plane are labelled by their Cartesian coordinates (x, z) (see Fig. 3.7).

[20] This is indeed the case; a quick numerical evaluation confirms it.

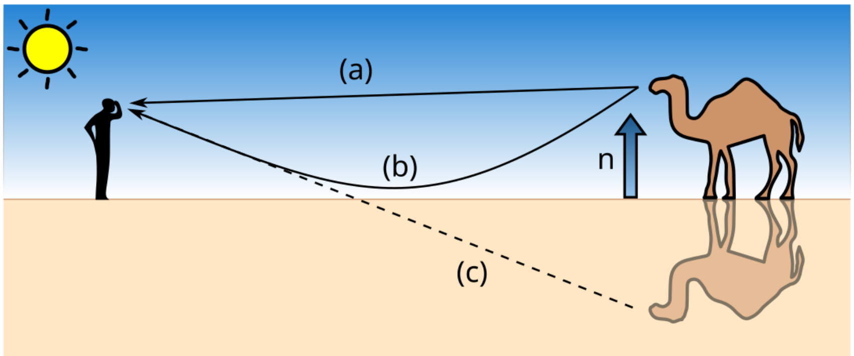

Figure 3.7: Diagram of a mirage (Wikipedia).

1. Show that the time taken for light to go from $(0, h)$ to (x_f, z_f) along a path $z(x)$ is:

$$T = \frac{1}{c} \int_0^{x_f} n(z(x)) \sqrt{1 + \left(z'(x)\right)^2} \, \mathrm{d}x \tag{3.155}$$

   where $z'(x)$ is the derivative of z with respect to x.

2. Since T is minimal, deduce from Beltrami's identity that the light path satisfies:

$$n(z(x))^2 = A\big(1 + (z'(x))^2\big), \tag{3.156}$$

   where A is a constant.

3. Suppose the ground is hot and the air above is cooler, so that the refractive index increases with z. We model:

$$n(z)^2 = n_0^2 + \alpha z. \tag{3.157}$$

   Show that:

$$A = \frac{n(h)^2}{1 + \tan^2 \theta}. \tag{3.158}$$

4. Show that the light path

$$z(x) = h + x \tan \theta + \frac{\alpha}{4A} x^2 \tag{3.159}$$

   is a solution to the problem (simply substitute to check).

5. Suppose an observer's eye is at point (L, H). Show that in general there exist two initial angles θ_1 and θ_2 allowing rays from $(0, h)$ to reach the observer.

6. Explain the mirage effect.

3.14.8 Fourth application: optical geodesics at sunset – linear model

Recall that in the previous section, we showed that the electron density follows:

$$N(z) = N_0 \left(\frac{T(z)}{T_0}\right)^k, \quad \text{where } T(z) = T_0 - \Gamma z. \tag{3.160}$$

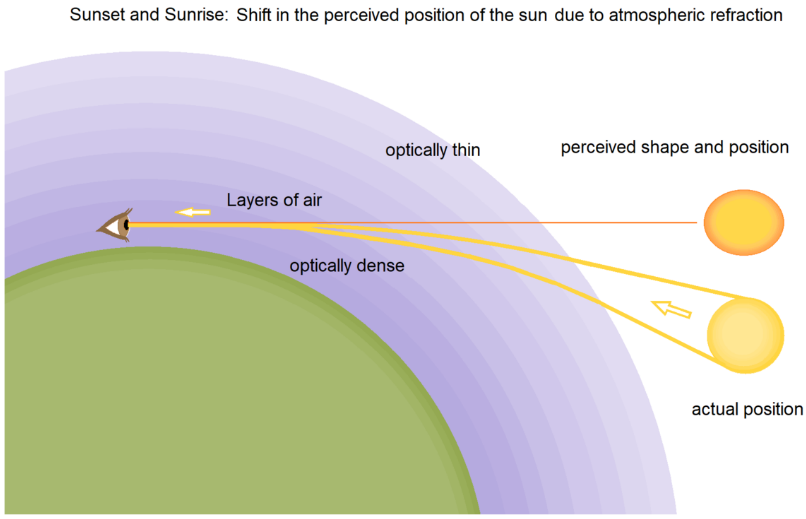

Figure 3.8: Schematic representation of light ray bending at sunset (Wikipedia).

1. Show that for z near zero, the Taylor expansion yields:

$$N(z) = N_0 \left(1 - \frac{k\Gamma}{T_0} z\right) + o(z). \tag{3.161}$$

   Conclude that $N(z)$ is approximately exponential, with characteristic height:

$$h := \frac{T_0}{k\Gamma}. \tag{3.162}$$

2. Deduce the approximate expression for the refractive index near the ground, showing its frequency dependence ω as:

$$n(z,\omega) \simeq n_0(\omega) - \alpha(\omega) z, \tag{3.163}$$

   with

$$n_0(\omega) := 1 + K(\omega), \quad \alpha(\omega) := \frac{k\Gamma}{T_0} K(\omega). \tag{3.164}$$

3. Using Fermat's principle, write the Lagrangian for the light ray path $z(x)$:

$$\mathcal{L}(z, z') = n(z,\omega)\sqrt{1 + (z')^2}. \tag{3.165}$$

   Show that since $\mathcal{L}$ does not explicitly depend on x, Beltrami's identity gives:

$$n(z,\omega)^2 = A(\omega)\left(1 + (z')^2\right), \tag{3.166}$$

   with constant $A(\omega) > 0$.

4. Using the linear expression of $n(z,\omega)$, write the differential equation for the trajectory:

$$(z')^2 = \frac{(n_0(\omega) - \alpha(\omega) z)^2}{A(\omega)} - 1. \tag{3.167}$$

   Show that the solution can be written explicitly as:

$$z(x) = h + x\tan\theta + \frac{\alpha(\omega)}{4A(\omega)} x^2, \tag{3.168}$$

with

$$A(\omega) = \frac{n_0(\omega)^2}{1 + \tan^2\theta}. \tag{3.169}$$

5. Qualitatively study the influence of frequency ω on the trajectory $z(x, \omega)$. Explain why:
   - Blue rays (high ω) are more curved than red ones.
   - The Sun remains visible even when it is geometrically below the horizon.
   - The Sun appears red at sunset.

3.14.9 Fifth application: Rayleigh scattering and the colour of the sky

1. Model an atmospheric molecule as an induced electric dipole $\mathbf{p}(t)$ under an incident electric field $\mathbf{E}(t)$ of light frequency ω. Write $\mathbf{p}(t)$ in terms of polarisability α_e and $\mathbf{E}(t)$.
2. Express the power radiated by an oscillating dipole in terms of its amplitude p_0, frequency ω, and observation angle θ.
3. Deduce the dependence of the scattered intensity on ω and α_e.
4. Using the relation between frequency ω and wavelength λ, show that scattered intensity varies as $1/\lambda^4$.
5. Explain why this $1/\lambda^4$ dependence leads to the blue colour of the sky.
6. Although violet light is scattered more than blue light, why does the sky appear blue rather than violet?
7. Based on this phenomenon, explain why sunlight appears reddish during sunrise or sunset.

3.15 Bose-Einstein Condensation [7] (PS) ★★★★

(Correction)

We consider a gas of identical bosonic particles with zero spin in a container of volume V in contact with a thermostat at temperature T. The particles do not interact with each other.

1. The gas is described in the grand canonical ensemble. We aim to express the average number of particles as a function of the temperature T and the chemical potential μ in the form of an integral, without attempting to evaluate it.
   (a) Write the expression of the average number $\langle n_\varepsilon \rangle$ of particles in a state of energy ε according to Bose-Einstein statistics, as a function of μ, T, and k_B.
   (b) Considering a gas in a cubic box of volume V with periodic boundary conditions, express the density of states $g(\varepsilon)$ in the approximation of a non-relativistic free gas of particles of mass m.
   (c) Deduce that the total average number of particles can be written as:
   $$\langle N \rangle = \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1} \mathrm{d}\varepsilon \tag{3.170}$$
   where $\beta = 1/(k_B T)$, then rewrite this expression as an integral depending on T, μ, and m, without solving it.
2. We now assume the system is closed and contains N particles. The chemical potential then becomes a function of temperature and particle density $\rho = N/V$. Using the previous result and the equivalence between canonical and grand canonical ensembles, show that $\mu(T)$ is given by:
   $$\rho = \left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^{x/\varphi(T)}-1} \mathrm{d}x, \tag{3.171}$$
   with $\varphi(T) = e^{\mu(T)/(k_BT)}$.
3. From equation (3.171), justify that $\mu(T)$ increases as the temperature decreases.
4. Recall why the chemical potential must be negative. Conclude that equation (3.171) can only be valid for $T \geq T_{\text{BE}}$, and determine the explicit expression of T_{BE}. Given:
   $$\int_0^{+\infty} \frac{x^{1/2}}{e^x - 1} \mathrm{d}x \simeq 2.612 \times \sqrt{\frac{\pi}{2}}. \tag{3.172}$$
5. For $T \leq T_{\text{BE}}$, $\mu(T) = 0$ and equation (3.171) is not satisfied. Identify the flaw in the reasoning from the previous questions.
6. To fix this issue, the population of the ground state, denoted N_0, is isolated. Justify that
   $$N = N_0 + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon}-1} \mathrm{d}\varepsilon. \tag{3.173}$$
   Then compute N_0 as a function of N, T, T_{BE}, and plot N_0/N as a function of T/T_{BE}. Comment.

7. Justify that for $T \leq T_{\text{BE}}$, the grand potential $\mathcal{J}$ is given by

$$\frac{\mathcal{J}}{k_B T} = -\ln(1 + N_0) + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \varepsilon^{1/2} \ln\left(1 - e^{-\beta\varepsilon}\right) \mathrm{d}\varepsilon. \tag{3.174}$$

What does this expression become in the thermodynamic limit? Then compute the pressure of the bosonic gas for $T \leq T_{\text{BE}}$ and $N \gg 1$, and comment on its dependencies. Given:

$$\int_0^{+\infty} \frac{x^{3/2}}{e^x - 1} \mathrm{d}x \simeq 1.341 \times \frac{3\sqrt{\pi}}{4}. \tag{3.175}$$

3.16 Decay Chain (FS) ★★★★★

(Correction)

We consider a radioactive decay chain formed by n isotopes denoted (N_k), which decay successively into one another ($N_1 \to N_2 \to \cdots \to N_n$), the last one being assumed stable. We denote by $N_k(t)$ the number of nuclei of type k at time $t \geq 0$. Each nucleus N_k is unstable for $k \in [\![1, n-1]\!]$ and has a radioactive decay constant $\lambda_k > 0$. The last isotope N_n is stable, which amounts to setting $\lambda_n = 0$.

3.16.1 Physical modeling of the decay chain

1. Justify that the functions $N_k(t)$ satisfy the differential system:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1, \qquad \frac{\mathrm{d}N_k}{\mathrm{d}t} = -\lambda_k N_k + \lambda_{k-1} N_{k-1} \quad \text{for } k \in [\![2, n]\!]. \tag{3.176}$$

2. Solve the case $n = 2$ with initial conditions $N_1(0) = N_0$, $N_2(0) = 0$. Sketch qualitatively the curves $N_1(t)$ and $N_2(t)$.

3. Show that the solution satisfies for all $t \geq 0$:

$$N_1(t) + N_2(t) = N_0. \tag{3.177}$$

   Physically interpret this: it corresponds here to conservation of matter in the system.

4. Discuss the time when the quantity $N_2(t)$ is maximal, and give its expression if $\lambda_1 \neq \lambda_2$.

3.16.2 Mathematical study of the differential system

Let $A \in \mathcal{M}_n(\mathbb{R})$ be the matrix defined by:

$$A = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & \lambda_2 & -\lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & 0 \end{pmatrix}. \tag{3.178}$$

Consider the vector system:

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} = A\mathbf{N}, \qquad \mathbf{N}(0) = \mathbf{N}_0 \in \mathbb{R}^n. \tag{3.179}$$

1. Show that A is diagonalizable over $\mathbb{R}$ if the λ_k (for $k \in [\![1, n-1]\!]$) are pairwise distinct. Give the eigenvalues.

2. Show that the system admits a unique global solution on $\mathbb{R}_+$ for any initial condition $\mathbf{N}_0$.

3. Let us define $E(t) = \|\mathbf{N}(t)\|^2$. Show that E is differentiable. We then aim to prove that E is a decreasing function.

(a) Show that

$$\forall x \in \mathbb{R}^n, \quad \langle x|Ax\rangle = \langle x|Sx\rangle, \quad \text{where } S = \frac{A + A^T}{2}. \tag{3.180}$$

(b) Show that

$$\langle x|Sx\rangle = -\frac{1}{2}\sum_k \lambda_k (x_k - x_{k+1})^2 - \frac{\lambda_1}{2}x_1. \tag{3.181}$$

(c) Discuss the sign of $E'(t)$. Deduce the stability of the system.

4. Assume that $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$. We define a norm satisfying:

$$\forall x \in \mathbb{R}^n, \quad \forall M \in \mathcal{M}_n(\mathbb{R}), \quad \|Mx\| \leq C \times \xi \|x\|, \tag{3.182}$$

where ξ satisfies $\xi > \mu_p \in \text{Sp}(M), \quad \forall p$.

Prove that:

$$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \leq Ce^{-\alpha t}\|\mathbf{N}_0\|, \quad \text{where } \mathbf{N}_\infty = (0, \ldots, 0, N_\infty). \tag{3.183}$$

5. Assume that $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$. We define a norm satisfying:

$$\forall x \in \mathbb{R}^n, \quad \forall M \in \mathcal{M}_n(\mathbb{R}), \quad \|Mx\| \leq C \times \xi \|x\|, \tag{3.184}$$

where ξ satisfies $\xi > \mu_p \in \text{Sp}(M), \quad \forall p$.

Prove that:

$$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \leq Ce^{-\alpha t}\|\mathbf{N}_0\|, \quad \text{where } \mathbf{N}_\infty = (0, \ldots, 0, N_\infty). \tag{3.185}$$

6. Verify that the system conserves the total amount of matter:

$$\sum_{k=1}^{n} N_k(t) = \sum_{k=1}^{n} N_k(0), \quad \forall t \geq 0. \tag{3.186}$$

3.17 From the Principle of Least Action to Einstein's Equations (SR, AM, GR) ★★★★★

(Solution)

3.17.1 From Classical Geometry to Lorentzian Geometry

1. **Geometry of a surface in $\mathbb{R}^3$.**

   Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface defined locally by a parametric chart:

$$\mathbf{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \tag{3.187}$$

   We denote $\partial_u \mathbf{X}$ and $\partial_v \mathbf{X}$ the tangent vectors to the surface, obtained as partial derivatives of the chart.

   (a) Show that the square of the infinitesimal length element can be written as:

$$\mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \tag{3.188}$$

   where:

$$E = \partial_u \mathbf{X} \cdot \partial_u \mathbf{X}, \quad F = \partial_u \mathbf{X} \cdot \partial_v \mathbf{X}, \quad G = \partial_v \mathbf{X} \cdot \partial_v \mathbf{X} \tag{3.189}$$

   (b) Show that the matrix:

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{3.190}$$

   defines such an inner product, and hence a Riemannian metric on the surface.

   (c) Compare with the flat case of the plane $\mathbb{R}^2$: show that in the canonical basis, we have $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. Discuss the role played by the parametric chart (u,v) in the local definition of the metric.

   (d) Show that g is symmetric, and thus diagonalizable. Deduce that its eigenvalues are real. These eigenvalues are called the **principal curvatures**.

   (e) We define the **Gaussian curvature** κ as the product of the two principal curvatures. Why is it a fundamental geometric invariant of a surface?

2. **Intrinsic Definition of a Metric.**

   Let $\mathcal{M}$ be a differentiable manifold of dimension n. A *Riemannian metric* is a tensor field $(g_p)_{p\in\mathcal{M}}$ that, at each point p, defines an inner product on the tangent space $T_p\mathcal{M}$.

   (a) Show that locally, in a chart (x^μ), the length element is written as:

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{3.191}$$

   (b) What is the relationship between this expression and the one obtained in (1)?

3. **Study of the Hyperbolic Metric on the Upper Half-Plane**

   Consider the upper half-plane:

$$\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \tag{3.192}$$

endowed with the so-called *hyperbolic* metric given by the length element:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{3.193}$$

(a) Verify that this expression defines a Riemannian metric (symmetric and positive definite) on $\mathbb{H}$.

(b) Compute the metric matrix g and its determinant $\det g$.

(c) Compute the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ associated with this metric.

(d) In two dimensions, κ is given by:

$$\kappa = -\frac{1}{\sqrt{\det g}} \left[\frac{\partial}{\partial x} \left(\frac{\Gamma^2_{12}\sqrt{\det g}}{g_{22}} \right) - \frac{\partial}{\partial y} \left(\frac{\Gamma^2_{11}\sqrt{\det g}}{g_{22}} \right) \right]. \tag{3.194}$$

(e) Conclude about the nature of the curvature of this metric (positive, zero, or negative).

(f) Using the principle of least action, we now compute the geodesics in this metric:

i. Write the Lagrangian $\mathcal{L}(y, y')$ associated with the length of a curve parametrized by x.

ii. Compute the derivative $\partial_{y'}\mathcal{L}$.

iii. Use the Beltrami's identity and show that, for $\lambda \neq 0$,

$$y\sqrt{1 + (y')^2} = \frac{1}{\lambda} \tag{3.195}$$

iv. Deduce that,

$$(x - x_0)^2 + y^2 = \frac{1}{\lambda^2} \tag{3.196}$$

v. Discuss the special case $\lambda = 0$. What geodesics are obtained in this case?

vi. (Bonus) By analyzing the form of the geodesics, what can we say about the symmetry of the hyperbolic plane?

(g) (Bonus) Provide a geometric interpretation of this metric and its connection to non-Euclidean geometry.

4. **Pseudo-Riemannian Manifolds.**

(a) Define a *Lorentzian* metric on a 4-dimensional manifold $\mathcal{M}$. What is its signature?

(b) Give the form of the Minkowski metric:

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1) \tag{3.197}$$

What do vectors with negative, zero, or positive norm represent?

5. **Curves and Geodesics.**

Let $\gamma(t) = \mathbf{X}(u(t), v(t))$ be a curve on a surface $\Sigma \subset \mathbb{R}^3$.

(a) Show that the length of the curve is:

$$L[\gamma] = \int_{t_1}^{t_2} \sqrt{g_{ij}\,\dot{x}^i\,\dot{x}^j}\,\mathrm{d}t \tag{3.198}$$

(b) Show that the curves which minimize this length satisfy the corresponding Euler-Lagrange equations.

3.17.2 Dynamics of Particles in Curved Spacetime

1. Consider a free particle of mass m in a spacetime endowed with a metric $g_{\mu\nu}(x)$. Its action is[21]:

$$S[x] = \frac{1}{2}\int g_{\mu\nu}(x)\,\dot{x}^\mu\,\dot{x}^\nu\,\mathrm{d}\tau \tag{3.199}$$

Show, using the principle of least action, that the equations of motion are:

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu = 0 \tag{3.200}$$

where the Christoffel symbols are:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{3.201}$$

2. Show that if one chooses a locally inertial frame (coordinates ξ^α), then:

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0 \tag{3.202}$$

and that via a change of variables, this implies the geodesic equation in the coordinates x^μ.

3.17.3 Curvature and the Einstein-Hilbert Action

We define the Levi-Civita connection as the unique torsion-free connection satisfying $\nabla_\lambda g_{\mu\nu} = 0$. Here are the formulas for the Riemann tensor, Ricci tensor, and scalar curvature:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{3.203}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu} \tag{3.204}$$

1. Show that if the metric is locally flat, then the Riemann tensor vanishes.

2. We seek a scalar, covariant action constructed solely from $g_{\mu\nu}$ and its derivatives up to order 2. Show that the Einstein-Hilbert action:

$$S[g] = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x \tag{3.205}$$

is (under these assumptions) the only possible one. Justify this uniqueness.

3.17.4 Principle of Least Action and Einstein's Equations

1. Let a field action be $S[\varphi] = \int \mathcal{L}(\varphi, \partial\varphi)\,\mathrm{d}^4x$. Its functional variation is defined by:

$$\delta S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}S[\varphi + \varepsilon\eta]\bigg|_{\varepsilon=0} \tag{3.206}$$

where $\eta(x)$ is an arbitrary compactly supported variation.

Show that the principle of least action implies the Euler-Lagrange equations:

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0 \tag{3.207}$$

[21] This can be shown. It is done (up to a constant factor) in Exercise 3.8.

2. Apply this principle to the Einstein-Hilbert action $S[g]$ to recover the gravitational field equations in vacuum:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{3.208}$$

3. Add a matter term:

$$S_{\text{total}} = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x + S_{\text{matter}}[g,\psi] \tag{3.209}$$

and define the energy-momentum tensor by:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{3.210}$$

(a) Justify this definition, explaining what is meant by the functional derivative with respect to the metric.

(b) Deduce the full Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.211}$$

3.18 A simple quantum particle near a black hole (QM, MA, GR, SP) ★★★★★

(Correction)

3.18.1 Proper time and relativistic gravitational potential

In the vacuum surrounding a spherically symmetric, uncharged black hole, the Schwarzschild metric reads:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{3.212}$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius.

1. Show that the proper time of a particle at rest at coordinate r is given by:

$$\mathrm{d}\tau = \sqrt{1 - \frac{r_s}{r}}\,\mathrm{d}t. \tag{3.213}$$

2. Assuming that the total energy is proportional to the ticking rate of a clock (as in quantum mechanics), deduce that the effective energy of a stationary particle is:

$$E(r) = mc^2 \sqrt{1 - \frac{r_s}{r}}. \tag{3.214}$$

3. Expand this expression as a series to second order in $\frac{r_s}{r} \ll 1$. Recover the Newtonian potential and a relativistic correction:

$$E(r) = mc^2 - \frac{GMm}{r} - \frac{1}{2}\frac{G^2M^2m}{c^2r^2} + o\left(\frac{r_s^2}{r^2}\right) \tag{3.215}$$

4. Identify the effective potential:

$$V_{\text{eff}}(r) = -\frac{GMm}{r} - \frac{1}{2}\frac{G^2M^2m}{c^2r^2}. \tag{3.216}$$

3.18.2 Expansion near the horizon

Set $r = r_s + x$ with $x \ll r_s$.

1. Express $V_{\text{eff}}(r)$ as a function of x, and expand to leading order. Show that:

$$V(x) = \frac{3}{2}\frac{GMm}{r_s^2}x + \text{const} + o(x). \tag{3.217}$$

2. Identify an effective linear potential:

$$V(x) = mg_{\text{eff}}x, \quad \text{with } g_{\text{eff}} = \frac{3}{2}\frac{GM}{r_s^2}. \tag{3.218}$$

and explain why this situation is analogous to that of a uniform gravitational field near a surface.

3.18.3 Quantum analysis of the linear potential

1. Write the Schrödinger equation in the local frame near the horizon:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + mg_{\text{eff}}x\psi(x) = E\psi(x). \tag{3.219}$$

2. Introduce the dimensionless variable $\xi = \frac{x-x_0}{x_c}$, and show that the equation becomes an Airy equation. Define x_0 and x_c.

3. Deduce that one solution is:

$$\psi(x) = \alpha \mathrm{Ai}(\xi) \tag{3.220}$$

with α a constant to be defined, and the function Ai also to be defined[22].

4. What condition must $\psi(0)$ satisfy? Observe that this induces a discrete sequence (E_n) of bound state energies.

3.18.4 Asymptotic Study of the Zeros of Ai

We consider the Airy function Ai, defined as the solution of the differential equation[23]:

$$\boxed{\psi''(x) = x\psi(x)} \tag{3.222}$$

with the decay condition at infinity:

$$\lim_{x\to+\infty} \psi(x) = 0. \tag{3.223}$$

We denote by Ai the $\mathscr{C}^\infty$ solution of (E) satisfying this property.

Study of the Airy Function

In this section, we show that the function $x \mapsto \mathrm{Ai}(x)$ vanishes infinitely many times on $\mathbb{R}_-^*$, but has no zeros on $\mathbb{R}_+$.

1. Show that Ai is of class $\mathscr{C}^\infty$ on $\mathbb{R}$, and that its zeros are isolated.

2. Show that if $x \geq 0$, then for all x, $\mathrm{Ai}''(x) = x\mathrm{Ai}(x) \geq 0$ if $\mathrm{Ai}(x) \geq 0$, and deduce that if Ai vanishes at a point $x_0 > 0$, then $\mathrm{Ai}(x) = 0$ for all $x > x_0$, which contradicts the decay towards 0. Deduce that $\mathrm{Ai}(x) > 0$ for all $x > 0$.

3. Assume, for the sake of contradiction, that the function Ai is strictly positive on $\mathbb{R}_-^*$.

   (a) Show that the derivative Ai' is strictly decreasing on $\mathbb{R}_-$ and that the following limit exists (possibly infinite):

$$\ell := \lim_{x\to-\infty} \mathrm{Ai}'(x) \in [-\infty, +\infty). \tag{3.224}$$

[22] In fact, another function is also a solution, but it is not considered in the WKB approximation. The other solution of the Airy equation reads:

$$\mathrm{Bi}(s) = \frac{1}{\pi}\int_{\mathbb{R}+} \exp\left(-\frac{t^3}{3} + st\right) + \sin\left(\frac{t^3}{3} + st\right)\mathrm{d}t. \tag{3.221}$$

[23] This section is considerably longer, more mathematical, and more technical than the previous ones; it can be omitted if one naturally accepts that the function $x \mapsto$ Ai admits a countable infinity of zeros on $\mathbb{R}_-^*$. However, for the skeptics and physicists who do not shy away from mathematics, this section is for you.

(b) Study the case where $\ell = \lim_{x\to-\infty} \mathrm{Ai}'(x)$ is a finite nonzero real.

i. Show that in this case, the improper integral

$$\int_{-\infty}^{0} \mathrm{Ai}''(t)\mathrm{d}t = \lim_{x\to-\infty} \big(\mathrm{Ai}'(0) - \mathrm{Ai}'(x)\big) \tag{3.225}$$

converges.

ii. Why does this convergence imply that $\mathrm{Ai}''(t) \to 0$ as $t \to -\infty$?

iii. Show that the convergence of the integral $\int_{\mathbb{R}_-^*} \mathrm{Ai}''$ implies that $\mathrm{Ai}(t) \underset{-\infty}{=} o(\frac{1}{|t|^{2+\varepsilon}})$, with $\varepsilon > 0$.

iv. Finally, show that Ai must satisfy $\mathrm{Ai}(t) \underset{-\infty}{=} \ell|t| + o(|t|)$. Deduce a contradiction, thus excluding the case $\ell \in \mathbb{R}^*$ finite.

(c) Study the case $\ell = 0$:

i. Assuming $\ell = 0$, show that for every $\varepsilon > 0$, there exists $M < 0$ such that for all $x < M$, $\mathrm{Ai}'(x) > -\varepsilon$.

ii. By integrating, deduce that

$$\mathrm{Ai}(x) = \mathrm{Ai}(M) + \int_{M}^{x} \mathrm{Ai}'(t)\mathrm{d}t > \mathrm{Ai}(M) - \varepsilon|x - M|, \tag{3.226}$$

and hence that $\mathrm{Ai}(x)$ tends to $+\infty$ as $x \to -\infty$.

iii. Deduce a contradiction.

(d) Study the case $\ell = -\infty$:

i. Using the intermediate value theorem, show that

$$\forall M < 0, \exists \chi < 0, \forall t < \chi, \quad \mathrm{Ai}(t) - \mathrm{Ai}(\chi) > M(t - \chi) \tag{3.227}$$

ii. Show that $\mathrm{Ai}''(t) \underset{-\infty}{=} -Mt^2 + \mathcal{O}(1)$.

iii. Show that this condition implies that $\mathrm{Ai}'(t) \underset{t\to-\infty}{\longrightarrow} +\infty$.

(e) Conclude.

Existence of a Countable Infinity of Zeros

Recall that the function $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ is a solution of the following differential equation:

$$\mathrm{Ai}''(x) + |x|\,\mathrm{Ai}(x) = 0 \quad \text{for all } x \leq 0, \tag{3.228}$$

For every $n \in \mathbb{N}^*$, we set $x_n := -n^2$ and $I_n := [x_n, x_n + \delta_n]$ with $\delta_n := \frac{2\pi}{n}$.

1. Show that for all $x \in I_n$, we have

$$|x| = n^2 + \varepsilon_n(x) \quad \text{with } \varepsilon_n(x) = \mathcal{O}\left(\frac{1}{n}\right). \tag{3.229}$$

2. Deduce that on I_n, the equation satisfied by Ai can be written as

$$\mathrm{Ai}''(x) + n^2\mathrm{Ai}(x) = f_n(x), \quad \text{with } f_n(x) = -\varepsilon_n(x)\mathrm{Ai}(x). \tag{3.230}$$

3. We wish to show that any function $y \in \mathscr{C}^2(I_n)$ written as a linear combination of the homogeneous basis

$$y_1(x) := \cos\big(n(x - x_n)\big), \quad y_2(x) := \sin\big(n(x - x_n)\big), \tag{3.231}$$

can be (not uniquely) expressed in the form

$$y(x) = u_n(x)y_1(x) + v_n(x)y_2(x), \tag{3.232}$$

for functions $u_n, v_n \in \mathscr{C}^1(I_n)$. We also wish to show that this method is equivalent to seeking a particular solution.

(a) Show that, for each $x \in I_n$, the linear map

$$\Phi_x : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u(x), v(x)) \mapsto \big(u(x)y_1(x)+v(x)y_2(x), \quad u(x)y_1'(x)+v(x)y_2'(x)\big), \tag{3.233}$$

is well-defined and linear.

(b) Show that for each $x \in I_n$, the linear map

$$\Phi_x : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, v) \mapsto \big(uy_1(x) + vy_2(x),\ uy_1'(x) + vy_2'(x)\big) \tag{3.234}$$

is an isomorphism.

(c) Consider now the linear map between function spaces

$$\Phi : \mathscr{C}^1(I_n)^2 \to \mathscr{C}^0(I_n), \quad (u, v) \mapsto uy_1 + vy_2, \tag{3.235}$$

where $uy_1 + vy_2$ is the pointwise-defined function on I_n.
Show that $\ker(\Phi)$ is nontrivial, i.e., there exists $(u, v) \neq (0, 0)$ in $\mathscr{C}^1(I_n)^2$ such that

$$u(x)y_1(x) + v(x)y_2(x) = 0 \quad \text{for all } x \in I_n. \tag{3.236}$$

Hint: Assume a proportionality relation between u, v and y_1, y_2 via an intermediate function α.

(d) Deduce that the set of representations of y is an affine class whose direction is generated by the pairs

$$\big(\sin(n(x - x_n)),\ -\cos(n(x - x_n))\big). \tag{3.237}$$

(e) Show that such a function α can be well-defined and of class $\mathscr{C}^1(I_n)$, for n sufficiently large. Interpret this kernel as the source of the non-uniqueness of the functions u_n, v_n for the representation of y.

(f) Deduce that imposing the auxiliary condition

$$u_n'(x)y_1(x) + v_n'(x)y_2(x) = 0 \tag{3.238}$$

is equivalent to choosing a section (a "complement") of the kernel, ensuring the uniqueness of the functions u_n, v_n. In other words, show that this condition fixes α.

(g) Conclude that the representation $y = u_ny_1 + v_ny_2$ with this condition is equivalent, in terms of solutions, to the classical form

$$y = y_p + Ay_1 + By_2, \tag{3.239}$$

where the particular part y_p is incorporated in the variation of coefficients u_n, v_n.

(h) By differentiating $y = u_n y_1 + v_n y_2$ and using the auxiliary condition, determine a simplified expression for $y'(x)$.

(i) By differentiating again and substituting into the equation $y'' + n^2 y = f_n$, establish the following linear system, solve it, and thus make explicit the functions u_n and v_n:

$$\begin{cases} u_n'(x)y_1(x) + v_n'(x)y_2(x) = 0, \\ u_n'(x)y_1'(x) + v_n'(x)y_2'(x) = f_n(x). \end{cases} \tag{3.240}$$

4. **Estimation of Variations on I_n**

   (a) Show that $\|f_n\|_{L^\infty(I_n)} = \mathcal{O}\big(\frac{1}{n}\big)$. Deduce that, for all $x \in I_n$,

   $$|u_n(x) - u_n(x_n)| \leq \mathcal{O}(n^{-2}). \tag{3.241}$$

   (b) Do the same for $v_n(x) - v_n(x_n)$. Deduce that u_n, v_n are almost constant on I_n.

5. **Final Approximation and Consequence**

   (a) Setting $a_n := u_n(x_n)$ and $b_n := v_n(x_n)$, show that for all $x \in I_n$,

   $$y(x) = a_n \cos\big(n(x - x_n)\big) + b_n \sin\big(n(x - x_n)\big) + \mathcal{O}\big(n^{-2}\big). \tag{3.242}$$

   Show that y converges uniformly to Ai.

   (b) Deduce that if $y = \mathrm{Ai}$ does not vanish on I_n, then the main combination $a_n \cos + b_n \sin$ has constant sign on I_n.

6. Show that y can also be written as

   $$y(x) = r_n \cos[n(x - x_n) - \varphi], \quad a_n \neq 0 \tag{3.243}$$

   with $\varphi = \arctan\big(\frac{b_n}{a_n}\big)$, $r_n = \sqrt{a_n^2 + b_n^2}$.

7. Deduce that any nontrivial linear combination of $\cos(n(x - x_n))$ and $\sin(n(x - x_n))$ changes sign on any interval of length strictly greater than $\frac{\pi}{n}$. Deduce that if the approximate solution remained of constant sign on I_n, then $|I_n| < \frac{\pi}{n}$.

8. Conclude.

We now denote the negative zeros of Ai by

$$c_k < c_{k+1} < 0, \forall k \quad \text{and we set } w_n := -c_n > 0. \tag{3.244}$$

Integral Representation and Qualitative Study

We admit that the Airy function admits a real integral representation over $\mathbb{R}$ given by:

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t. \tag{3.245}$$

1. We aim to show that this integral converges for all $x \in \mathbb{R}$ and defines a continuous function.

   (a) Show that studying Ai reduces to studying the following integral,

   $$h(x) = \int_0^{\infty} e^{i f_x(t)} \mathrm{d}t \tag{3.246}$$

(b) Quickly analyze f_x. Compute the minimum $t_0(x)$.

(c) Let R be to be determined. By splitting the integral and cleverly expressing $e^{if_x(t)}$ in terms of its derivative with respect to t, show that h is well-defined for all $x \in \mathbb{R}$.

(d) Let $x, y \in \mathbb{R}$. Show that

$$|h(y) - h(x)| \underset{y \to x}{\longrightarrow} 0 \tag{3.247}$$

(e) Using a similar method, show that h is of class $\mathscr{C}^1$.

2. To rigorously show that $\mathrm{Ai}(x) \to 0$ as $x \to +\infty$, we will use the method of steepest descents. We replace the previous single question with the following steps (note $f_x(t) = xt + \frac{t^3}{3}$) :

(a) Show that $t \mapsto e^{if_x(t)}$ is an entire function of the complex variable t. Conclude that the integration contour $\mathbb{R}$ can be deformed in the complex plane without encountering singularities.

(b) Solve $f_x'(t) = 0$ in $\mathbb{C}$. Verify that the saddle points are

$$t_\star = \pm i\sqrt{x}, \tag{3.248}$$

and determine which one(s) satisfy $\mathrm{Re}(if_x(t_\star)) < 0$.

(c) Explicitly construct a descent contour Γ_x (arising from a deformation of $\mathbb{R}$) passing through $t_\star = i\sqrt{x}$ in the direction of steepest descent. Parametrize a neighborhood of $t_\star$ by

$$t = t_\star + z, \qquad z = u\,x^{-1/4}, \quad u \in \mathbb{R}. \tag{3.249}$$

(d) Expand f_x in a Taylor series around $t_\star$. Show that, for $t = t_\star + z$,

$$f_x(t) = f_x(t_\star) + \tfrac{1}{2} f_x''(t_\star)\, z^2 + \tfrac{1}{6} f_x^{(3)}(\xi)\, z^3, \tag{3.250}$$

with $f_x''(t_\star) = 2t_\star$ and $f_x^{(3)} \equiv 2$. Multiplying by i and after the change $z = ux^{-1/4}$, obtain the identity

$$if_x(t) = if_x(t_\star) - u^2 + i\,O\big(u^3 x^{-3/4}\big), \qquad (x \to \infty), \tag{3.251}$$

uniformly for $|u| \le M$ fixed.

(e) Show the local uniform estimate: for all $M > 0$,

$$e^{\,if_x(t)} = e^{\,if_x(t_\star)} e^{-u^2}\big(1 + o(1)\big) \qquad \text{uniformly for } |u| \le M, \tag{3.252}$$

and deduce, via the change of variable $dt = x^{-1/4}\,du$ and dominated convergence,

$$\int_{|u| \le M} e^{\,if_x(t)}\,dt = e^{\,if_x(t_\star)} x^{-1/4} \Big(\int_{|u| \le M} e^{-u^2}\,du \Big)\,(1 + o(1)). \tag{3.253}$$

(f) Control the tail contribution $|u| > M$: show that, for fixed large M,

$$\left| \int_{|u| > M} e^{\,if_x(t)}\,dt \right| \le x^{-1/4} \int_{|u| > M} e^{-u^2}\,du, \tag{3.254}$$

and that this bound can be made arbitrarily small independently of x.

(g) Show that the remaining contribution on $\Gamma_x \setminus (\text{neighborhood of } t_\star)$ is negligible compared to $x^{-1/4} e^{-\frac{2}{3}x^{3/2}}$: on this portion of Γ_x we have $\mathrm{Re}(if_x(t)) \le -cx^{3/2}$ for some constant $c > 0$, yielding an exponential bound.

(h) Explicitly compute $if_x(t_\star)$ for $t_\star = i\sqrt{x}$ and conclude:

$$if_x(i\sqrt{x}) = -\tfrac{2}{3}x^{3/2}, \tag{3.255}$$

then gather the estimates to obtain the asymptotic

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}}\, x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \quad (x \to +\infty), \tag{3.256}$$

in particular $\mathrm{Ai}(x) \to 0$.

Asymptotic Approximation of $\mathrm{Ai}(-x)$ for $x \to +\infty$

We recall the definition

$$f_x(t) := \frac{t^3}{3} - xt. \tag{3.257}$$

1. Compute the stationary point t_0 of the phase f_x on $[0, +\infty[$ and show that it is unique.
2. Show that for any $\delta > 0$, there exists a constant $c_\delta > 0$, independent of x, such that for all t satisfying

$$|t - t_0| \geq \delta, \tag{3.258}$$

we have

$$|f_x'(t)| = |t^2 - x| \geq c_\delta. \tag{3.259}$$

3. Let a, b satisfy $0 \leq a < b$ and $\min_{t\in[a,b]} |f_x'(t)| \geq c_\delta > 0$. Show that, by integration by parts,

$$\left| \int_a^b e^{if_x(t)} \mathrm{d}t \right| \leq \frac{2}{c_\delta} + (b-a) \sup_{t\in[a,b]} \left| \frac{f_x''(t)}{f_x'(t)^2} \right|. \tag{3.260}$$

4. Show that for any fixed $\delta > 0$, the contribution outside a neighborhood of $t_0 = \sqrt{x}$ is $o(x^{-1/4})$ as $x \to +\infty$, i.e.,

$$\int_{\substack{t\geq 0\\ |t-t_0|\geq\delta}} e^{if_x(t)} \mathrm{d}t = o\left(x^{-1/4}\right). \tag{3.261}$$

To do so, perform successive integrations by parts using the uniform lower bound of $|f_x'(t)|$ outside this neighborhood, and justify why this contribution becomes negligible compared to the one from the vicinity of t_0.

5. Show that for t close to t_0, we have the expansion

$$f_x(t) = f_x(t_0) + \frac{f_x''(t_0)}{2}(t-t_0)^2 + R_x(t), \tag{3.262}$$

with

$$R_x(t) = \frac{f_x^{(3)}(\xi)}{6}(t-t_0)^3, \tag{3.263}$$

where ξ lies between t and t_0. Compute $f_x''(t_0)$ and $f_x^{(3)}(t)$, and show that there exists $C > 0$ such that

$$|R_x(t)| \leq C|t-t_0|^3 \quad \text{for } |t-t_0| \leq \delta. \tag{3.264}$$

6. Show that for all t,

$$|e^{iR_x(t)} - 1| \leq |R_x(t)|. \tag{3.265}$$

7. Define

$$\varepsilon_x := \int_{|t-t_0|\leq\delta} e^{if_x(t_0)} e^{i\frac{f_x''(t_0)}{2}(t-t_0)^2} \left(e^{iR_x(t)} - 1\right) \mathrm{d}t, \tag{3.266}$$

and show that

$$|\varepsilon_x| \leq C'\delta^4. \tag{3.267}$$

8. Perform the change of variable

$$s = (t - t_0)\sqrt{\frac{|f_x''(t_0)|}{2}}, \tag{3.268}$$

and show that

$$\int_{|t-t_0|\leq\delta} e^{i\frac{f_x''(t_0)}{2}(t-t_0)^2} \mathrm{d}t = \sqrt{\frac{2}{|f_x''(t_0)|}} \int_{|s|\leq S_x} e^{i\sigma s^2} \mathrm{d}s, \tag{3.269}$$

where $\sigma = \mathrm{sign}(f_x''(t_0))$ and $S_x = \delta\sqrt{\frac{|f_x''(t_0)|}{2}}$.

9. Show that $S_x \to +\infty$ as $x \to +\infty$.

10. Compute the complete Fresnel integral:

$$\int_{-\infty}^{+\infty} e^{i\sigma s^2} \mathrm{d}s = \sqrt{\pi} e^{i\sigma\pi/4}. \tag{3.270}$$

11. Combine all previous approximations to obtain the asymptotic equivalence

$$\mathrm{Ai}(-x) \sim \frac{C}{x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \tag{3.271}$$

specifying the constant C (or noting that it depends on the exact normalization of Ai).

Asymptotic Expansion of the Sequence (w_n)

1. Deduce that the zeros w_n of $\mathrm{Ai}(-x)$ satisfy asymptotically:

$$\cos\left(\frac{2}{3}w_n^{3/2} - \frac{\pi}{4}\right) = 0. \tag{3.272}$$

2. Show that this equation implies:

$$\frac{2}{3}w_n^{3/2} = \left(n - \frac{1}{2}\right)\pi + \frac{\pi}{4}. \tag{3.273}$$

3. Deduce that:

$$w_n \sim \left(\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right)^{2/3}. \tag{3.274}$$

Asymptotic Expansion of the Zeros w_n to Order $\mathcal{O}(1/n^2)$

Recall that the negative zeros of $\mathrm{Ai}(-x)$, denoted w_n, satisfy asymptotically:

$$\frac{2}{3} w_n^{3/2} = \pi \left(n - \frac{1}{4} \right). \tag{3.275}$$

Let $\alpha_n := \left(\frac{3\pi}{2}(n - \frac{1}{4})\right)^{2/3}$ denote the first approximation of w_n.

1. Set $w_n = \alpha_n + \beta_n$, with $\beta_n = o(\alpha_n)$, and expand $(\alpha_n + \varepsilon_n)^{3/2}$ in a Taylor series to second order around α_n.
2. Substitute this expansion into the equation satisfied by w_n, and deduce an explicit asymptotic expression for ε_n to order $\mathcal{O}(\alpha_n^{-1})$.
3. Deduce an expansion of w_n to order $\mathcal{O}(1/n^2)$, also expanding α_n^{-1} in terms of n.

Energy of Bound States

1. Deduce a series expansion of (E_n) to order $\mathcal{O}(1/n^2)$.
2. For M being the mass of the Sun, estimate the energy levels. Interpret the results.

3.18.5 Horizon, Absorption, and Decoherence

In the previous section, we obtained stationary bound states described by Airy functions, $\psi(x) = \alpha\,\mathrm{Ai}(\xi)$, with decay towards zero as $x \to +\infty$ and a damped oscillatory behavior as $x \to -\infty$.

However, this idealized model assumes perfect confinement, which is only an approximation. In reality, the particle can escape toward the region $x \to -\infty$, corresponding to the interior of the black hole, leading to probability loss and temporal decay of the quantum state.

1. Explain why the decay of $\mathrm{Ai}(\xi)$ to zero on the left (for $x \to -\infty$) does not prevent a slow leakage of the particle into this region. One may rely on the asymptotic approximation for $x \to -\infty$:
$$\mathrm{Ai}(x) \sim \frac{C}{|x|^{1/4}} \cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4} \right), \tag{3.276}$$
which reflects a damped oscillatory, non-confined behavior.
2. Considering this leakage, justify that the quantum state is not strictly stationary and that the norm of the wavefunction $\Psi(x,t)$ decreases over time.
3. To model this decay, we introduce a complex energy:
$$E \mapsto E - i\frac{\Gamma}{2}, \quad \Gamma > 0, \tag{3.277}$$
and write the non-Hermitian time-dependent Schrödinger equation:
$$i\hbar \frac{\partial}{\partial t}\Psi = \left(E - i\frac{\Gamma}{2} \right) \Psi. \tag{3.278}$$
Show that the temporal solution is then:
$$\Psi(t) = \Psi(0) e^{-i\frac{E}{\hbar}t} e^{-\frac{\Gamma}{2\hbar}t}, \tag{3.279}$$
and interpret the decaying exponential factor as a norm loss associated with the leakage.

4. Discuss the analogy with radioactive decay, noting that the quantity $\tau = \frac{\hbar}{\Gamma}$ corresponds to the mean lifetime of the quasi-bound state near the horizon.

3.18.6 Opening: Hawking Radiation and Thermal Temperature

We aim to rigorously recover the temperature of the Hawking radiation associated with a Schwarzschild black hole of mass M [24].

1. **Action of a particle in the Schwarzschild metric**

   Recall the Schwarzschild metric, valid for $r > r_s = \frac{2GM}{c^2}$:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{3.280}$$

   (a) Give the expression of the relativistic Lagrangian for a particle of mass m in this metric, assuming purely radial motion ($\theta = \pi/2$, $\dot{\theta} = \dot{\varphi} = 0$).

   (b) Show that the classical action of the particle can be written as:

$$S = -mc \int \mathrm{d}s = -mc \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\mathrm{d}\lambda. \tag{3.281}$$

   (c) Assuming the energy E of the particle is a constant of motion, express the radial velocity $\dot{r}$ in terms of E and r[25].

2. **Effective form of the radial action**

   We want to calculate the probability that a virtual particle generated in the vacuum just near the horizon tunnels through.

   (a) Derive the Hamilton-Jacobi equation in the relativistic framework:

$$\partial_\mu S \partial^\mu S + mc^2 = 0 \tag{3.282}$$

   (b) Show that the radial action of the particle, with energy E, can be written as:

$$S_r = \int p_r \mathrm{d}r, \tag{3.283}$$

   where p_r is the radial momentum obtained from the Lagrangian.

   (c) Show that this momentum takes the form:

$$p_r = \frac{1}{c}\frac{E}{f(r)} = \frac{1}{c}\frac{E}{1 - \frac{r_s}{r}}, \tag{3.284}$$

   considering a radial trajectory at fixed energy, and assuming the rest-mass energy mc^2 is negligible compared to E.

3. **Tunneling effect and complex integral**

[24] Relying, of course, on the semi-classical analogy between tunneling effect and the probability of spontaneous emission through a gravitational barrier. No QFT knowledge is required: only the ideas of relativistic classical action and the variational principle will be used.

[25] Recall that continuous time symmetry implies energy conservation (Noether).

(a) Consider integrating over $I = [r_s - \varepsilon, r_s + \varepsilon], \varepsilon > 0$. The integral $S_r = \int_I p_r \mathrm{d}r$ diverges at $r = r_s$ because $f(r)$ vanishes. Show that this divergence can be avoided by integrating in the complex plane, deforming the integration path around the pole $r = r_s$.

(b) Deduce that

$$S = \oint_\gamma p_r \mathrm{d}r = 2\pi i \frac{E r_s}{c} \tag{3.285}$$

(c) Relating S to the quantum mechanical (WKB) tunneling effect, deduce the tunneling probability in this case,

$$\mathbb{P}(E) = \exp\left(-\frac{8\pi G M E}{\hbar c^3}\right). \tag{3.286}$$

4. **Identification with a thermal law**

Comparing $\mathbb{P}(E)$ with a Boltzmann-type distribution, rigorously identify the Hawking temperature:

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}. \tag{3.287}$$

5. **Energy and entropy of the black hole**

Assuming the emission of radiation can be treated as a reversible thermodynamic process, we write[26]:

$$\mathrm{d}S = \frac{\mathrm{d}M}{T_H}. \tag{3.288}$$

(a) Integrate this equation to obtain an explicit expression for the entropy:

$$S = \int \frac{\mathrm{d}M}{T_H(M)} = \int \frac{8\pi G k_B M}{\hbar c^3} \mathrm{d}M = \frac{4\pi G k_B}{\hbar c^3} M^2 + \text{constant}. \tag{3.289}$$

(b) Using the fact that the horizon area is $\Sigma = 4\pi r_s^2 = \frac{16\pi G^2 M^2}{c^4}$, deduce:

$$S = \frac{k_B c}{4G\hbar} \Sigma. \tag{3.290}$$

6. **Physical discussion**

(a) Why is this entropy proportional to the area rather than the volume?

(b) What questions does this formula raise about the microscopic nature of black holes and quantum gravity?

7. **Evaporation time of a black hole**

The evaporation time of a black hole can be estimated using the Stefan-Boltzmann law. The energy radiated by a body of radius R and temperature T is, with σ the Stefan-Boltzmann constant (c.f. 3.6):

$$L = 4\pi R^2 \sigma T^4, \tag{3.291}$$

(a) For a Schwarzschild black hole, show that

$$\frac{\mathrm{d}(Mc^2)}{\mathrm{d}t} = -4\pi r_s^2 \sigma T_H^4. \tag{3.292}$$

[26]Here we change notation: S now denotes entropy.

(b) Replacing r_s and T_H, show that the differential equation for the mass evolution is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{15360\pi}\frac{c^4\hbar}{G^2}\frac{1}{M^2}. \tag{3.293}$$

(c) By integrating this equation, show that the total evaporation time of a black hole of mass M is

$$t_e = 5120\pi\frac{G^2}{c^4\hbar}M^3. \tag{3.294}$$

(d) In SI units, estimate the order of magnitude of the evaporation time of a solar-mass black hole $M_\odot$.

(e) Compare this time to the current age of the universe ($\sim$ 13.8×10^9 years). What conclusion can be drawn regarding the observation of the evaporation of stellar black holes?

(f) Estimate the evaporation time for a hypothetical primordial black hole of mass 10^{12} kg. Would this evaporation be observable today?

Chapter 4

Exercise Solutions

As you may notice, not **all** exercises have been solved yet. Unsolved exercises are marked with the symbol △. The remaining solutions will be added progressively. If you would like to submit a solution, please send it to the following email address in LaTeX format:
ryanartero2005@gmail.com.
Moreover, you can return to the exercise you were working on by clicking on its title, either at the top of the page or at the beginning of the exercise.

4.1 Two-Body Problem

4.1.1 Center of Mass

We denote by $\mathbf{r}_1, \mathbf{r}_2$ the position vectors of the electron and the nucleus with respect to an arbitrary reference frame, and by $\mathbf{v}_1, \mathbf{v}_2$ the corresponding velocities.

1. $\mathcal{L} = \frac{1}{2}(m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2) - \frac{\vartheta^2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}$.

2.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \implies \mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \tag{4.1}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \implies \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \tag{4.2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{4.3}$$

$$\implies \mathcal{L} = \frac{1}{2}(m_1 + m_2)\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2 - \frac{\vartheta^2}{r} = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v}) \tag{4.4}$$

3. The potential is central for the center of mass. This implies that $\mathbf{J}$ is a conserved quantity.

In the following, we focus exclusively on the internal motion described by $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to $\mathbf{J}$.

4.1.2 Integration of the Equations of Motion

1. The expression for the kinetic energy in polar coordinates in $\mathbb{R}^2$ is:

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2), \tag{4.5}$$

which gives the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}, \quad \text{with } k = \vartheta^2. \tag{4.6}$$

The Euler-Lagrange equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{k}{r^2} = 0, \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu r^2\dot{\theta}) = 0. \tag{4.8}$$

Conjugate momenta are given by:

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r}, \qquad p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \mu r^2\dot{\theta}. \tag{4.9}$$

The Hamiltonian reads:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \mathcal{L} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r}. \tag{4.10}$$

Hamilton's equations are then:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{\mu r^3} - \frac{k}{r^2}, \tag{4.11}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0. \tag{4.12}$$

p_θ is a conserved quantity (since θ is a cyclic variable); thus, $p_\theta = \mu r^2\dot{\theta}$ is constant — the angular momentum J, fixed by the initial conditions.

Indeed, $\mathbf{J} = \mu\mathbf{r} \times \dot{\mathbf{r}} = \mu r\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = \mu r^2\dot{\theta} = p_\theta$.

The first integral of energy is:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}. \tag{4.13}$$

By differentiating $p_r = \mu\dot{r}$ and substituting:

$$\dot{p}_r = \mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}, \tag{4.14}$$

we recover the radial equation of motion:

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}. \tag{4.15}$$

The first term on the right-hand side is the centrifugal force, the second is the attractive Coulomb force.

2. To eliminate time, we differentiate the composite function $r(\theta(t))$:
   Let $r'(\theta) = \frac{\mathrm{d}r}{\mathrm{d}\theta}$ and $r''(\theta) = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2}$. Using $p_\theta = \mu r^2 \dot{\theta} = J$, we get:
$$\dot{\theta} = \frac{J}{\mu r^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{4.16}$$
   Thus:
$$\dot{r} = r'\frac{J}{\mu r^2}, \qquad \ddot{r} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r'\frac{J}{\mu r^2}\right). \tag{4.17}$$
   Setting $u = \frac{1}{r}$, we obtain:
$$\dot{r} = -\frac{J}{\mu}u', \qquad \ddot{r} = -\frac{J^2}{\mu^2}(u'' + u), \tag{4.18}$$
   and substitution into (7.25) gives:
$$-\frac{J^2}{\mu^2}(u'' + u) = \frac{J^2}{\mu}u^3 - \frac{k}{\mu}u^2. \tag{4.19}$$
   Multiplying both sides by $-\frac{\mu^2}{J^2}$ yields:
$$u'' + u = \frac{\mu k}{J^2}. \tag{4.20}$$

3. The differential equation in $u(\theta)$:
$$u'' + u = \frac{\mu k}{J^2} \tag{4.21}$$
   has the general solution:
$$u(\theta) = A\cos(\theta + \varphi) + \frac{\mu k}{J^2}, \tag{4.22}$$
   hence:
$$r(\theta) = \frac{1}{A\cos(\theta + \varphi) + \frac{\mu k}{J^2}}. \tag{4.23}$$
   One can always choose the polar axis so that $r(\theta)$ is extremal at $\theta = 0$ (or π), which gives $\varphi = 0$:
$$r(\theta) = \frac{1}{\frac{\mu k}{J^2}(1 + \varepsilon\cos\theta)}, \tag{4.24}$$
   where $\varepsilon = \dfrac{AJ^2}{\mu k}$ is the eccentricity.
   The constant A (or ε) is determined by the initial conditions or by the energy:
$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}. \tag{4.25}$$
   Using $r(\theta)$ and $J = \mu r^2\dot{\theta}$, one can express E as a function of ε:
$$\varepsilon^2 = 1 + \frac{2EJ^2}{\mu k^2}. \tag{4.26}$$

4. Equation (4.24) defines a family of curves called **conic sections** (intersections of a cone with a plane). Three subfamilies are distinguished according to the value of ε:

- If $\varepsilon < 1$, the trajectory is an **ellipse**, closed, corresponding to energy $E < 0$: a bound and periodic motion (in particular, $\varepsilon = 0$ gives a circle).
- If $\varepsilon = 1$, the trajectory is a **parabola**: a limiting case $E = 0$ separating bound and unbound motions.
- If $\varepsilon > 1$, the denominator in 4.24 can vanish for some angle $\theta_\infty = \arccos\left(-\frac{1}{\varepsilon}\right)$: the trajectory is a (open) **hyperbola** with asymptotes; $E > 0$ corresponds to a particle arriving from infinity with nonzero initial velocity.

In all cases, the origin (center of force) is one of the foci of the conic.

4.1.3 Bohr Quantization

We consider the case $E < 0$ (bound states). The Bohr–Sommerfeld quantization conditions are:

$$J_\theta := \oint p_\theta \,\mathrm{d}\theta = n_\theta h, \qquad J_r := \oint p_r \,\mathrm{d}r = n_r h, \qquad n_\theta, n_r \in \mathbb{Z}. \tag{4.27}$$

1. For planar motion in a central potential, the angular momentum p_θ is conserved:

$$p_\theta = J. \tag{4.28}$$

Therefore:

$$J_\theta = \int_0^{2\pi} p_\theta \,\mathrm{d}\theta = 2\pi J \quad \Rightarrow \quad J = n_\theta \hbar. \tag{4.29}$$

Since $n_\theta = 0$ would correspond to a rectilinear trajectory passing through the center (which is excluded here), we have:

$$n_\theta \in \mathbb{N}^*. \tag{4.30}$$

2. For J_r:

$$p_r = \mu \dot{r} = \frac{J}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{4.31}$$

since $\dot{\theta} = \frac{J}{\mu r^2}$. Using the conic equation

$$r(\theta) = \frac{p}{1 + \varepsilon \cos\theta}, \qquad p = \frac{J^2}{\mu \vartheta^2}, \tag{4.32}$$

we obtain:

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{p\,\varepsilon \sin\theta}{(1 + \varepsilon\cos\theta)^2}. \tag{4.33}$$

Thus:

$$p_r = \frac{J}{r^2} \cdot \frac{p\,\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2} = \frac{J\,\varepsilon\sin\theta}{p}. \tag{4.34}$$

Substituting $p = \frac{J^2}{\mu\vartheta^2}$ gives:

$$p_r = \frac{\mu\vartheta^2\,\varepsilon\sin\theta}{J}. \tag{4.35}$$

For J_r:

$$J_r = 2\int_{r_{\min}}^{r_{\max}} p_r \,\mathrm{d}r. \tag{4.36}$$

Changing variables $r \mapsto \theta$ over half of the orbit ($0 \to \pi$):

$$\mathrm{d}r = \frac{\mathrm{d}r}{\mathrm{d}\theta}\,\mathrm{d}\theta = \frac{p\,\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta, \tag{4.37}$$

so that:

$$p_r\,\mathrm{d}r = \frac{\mu\vartheta^2\,\varepsilon\sin\theta}{J}\cdot\frac{p\,\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta = \frac{\mu\vartheta^2\,p}{J}\cdot\frac{\varepsilon^2\sin^2\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta. \tag{4.38}$$

Since $\frac{\mu\vartheta^2\,p}{J} = J$ (using $p = \frac{J^2}{\mu\vartheta^2}$), we find:

$$p_r\,\mathrm{d}r = J\,\frac{\varepsilon^2\sin^2\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta. \tag{4.39}$$

As J_r corresponds to a complete radial cycle (forth and back), we integrate from 0 to π and multiply by 2:

$$J_r = 2J\,\varepsilon^2\int_0^\pi \frac{\sin^2\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta. \tag{4.40}$$

The condition $J_r = n_r h$ becomes:

$$2J\,\varepsilon^2\int_0^\pi \frac{\sin^2\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta = n_r h. \tag{4.41}$$

The integral can be evaluated by integration by parts or using the given formula:

$$\int_0^\pi \frac{\sin^2\theta}{(1+\varepsilon\cos\theta)^2}\,\mathrm{d}\theta = \frac{\pi}{\varepsilon^2}\left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1\right). \tag{4.42}$$

Hence:

$$2\pi J\left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1\right) = n_r h. \tag{4.43}$$

Using $2\pi J = n_\theta h$:

$$n_\theta\left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1\right) = n_r. \tag{4.44}$$

3. From the equation relating ε to the energy and angular momentum:

$$1-\varepsilon^2 = -\frac{2EJ^2}{\mu\vartheta^4}. \tag{4.45}$$

Substituting $J = n_\theta\hbar$ and using the above relation between ε, n_r, and n_θ, we obtain:

$$1-\varepsilon^2 = \left(\frac{n_\theta}{n}\right)^2, \qquad n := n_r + n_\theta. \tag{4.46}$$

We then find:

$$E_n = -\frac{\mu\vartheta^4}{2\hbar^2 n^2}, \qquad n \in \mathbb{N}^*. \tag{4.47}$$

4.2 Rutherford Scattering Cross-Section

4.2.1 Deflection of a Charged Particle by an Atomic Nucleus

We work in a polar coordinate system (r, φ) in the plane of motion.

1. **Angular Momentum:** The angular momentum in polar coordinates is:
$$J = mr^2\dot{\varphi}. \tag{4.48}$$
At past infinity, the particle has speed v_0 and an impact parameter b. The angular momentum is then:
$$J = -mbv_0. \tag{4.49}$$
The negative sign comes from the fact that φ decreases with time.

2. **Equation of Motion:** The central repulsive force is given by:
$$\mathbf{F} = \frac{C}{r^2}\hat{\mathbf{r}}, \quad \text{where } C = \frac{qQ}{4\pi\varepsilon_0}. \tag{4.50}$$
We decompose $\mathbf{v} = \dot{\mathbf{r}}$ into two components. Projecting onto the direction perpendicular to the polar axis, we find:
$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi. \tag{4.51}$$

3. **Deflection Angle θ:** Multiplying the equation by $\mathrm{d}t$ and changing variables, we use:
$$r^2\dot{\varphi} = \frac{J}{m} \Rightarrow \mathrm{d}t = \frac{mr^2}{J}\mathrm{d}\varphi. \tag{4.52}$$
Integrating from $t = -\infty$ to $t = +\infty$:
$$v_0 \sin\theta = \int \dot{v}_\perp \mathrm{d}t = \frac{C}{J}(\cos\theta + 1). \tag{4.53}$$

4. **Relation to Kinetic Energy:** The initial energy is $E_0 = \frac{1}{2}mv_0^2$, so:
$$\tan\left(\frac{\theta}{2}\right) = \frac{C}{2E_0 b}. \tag{4.54}$$

4.2.2 Rutherford Scattering Cross-Section

1. **Expression for the Differential Cross-Section:** The general definition is $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$.

2. **Using $\tan(\theta/2)$:** With:
$$b = \frac{C}{2E_0}\cot\left(\frac{\theta}{2}\right), \quad \frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{C}{4E_0}\frac{1}{\sin^2(\theta/2)}, \implies \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{C}{4E_0}\right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{4.55}$$

3. **Limit of the Model:** For $\theta \to 0$, we have $\sin(\theta/2) \to 0$ so $\mathrm{d}\sigma/\mathrm{d}\Omega \to \infty$. The integral over $\theta \in [0, \pi]$ diverges: the total cross-section is infinite. This reflects the infinite range of the Coulomb interaction.

4. **Experimental Interpretation:** This model explains Rutherford's experimental results: alpha particles can be strongly deflected. This implies the existence of a highly concentrated atomic nucleus, as such deflection requires a very intense field in a very localized region[1].

[1] By introducing the minimum approach distance $a_{\min}$ for a head-on collision ($b = 0$), we have:
$$a_{\min} = \frac{C}{E_0}. \tag{4.56}$$

4.3 Cherenkov Effect

1. The energy of a photon is given by the standard relation:

$$E_\gamma = h\nu \tag{4.58}$$

In a medium with refractive index n, the phase velocity of light is reduced to c/n, and the associated wave vector is:

$$k = \frac{2\pi n\nu}{c} \tag{4.59}$$

The momentum of the photon in this medium is therefore:

$$p_\gamma = \hbar k = \frac{hn\nu}{c} \tag{4.60}$$

Thus, we obtain the desired relation:

$$\boxed{p_\gamma = \frac{hn\nu}{c}} \tag{4.61}$$

Combining this with the expression for the energy $E_\gamma = h\nu$, we deduce:

$$\boxed{p_\gamma = \frac{n}{c}E_\gamma} \tag{4.62}$$

2. The components of momentum are:

$$p = p'\cos\varphi + p_z\cos\theta, \quad 0 = -p'\sin\varphi + p_z\sin\theta. \tag{4.63}$$

3. We have:

$$p_z^2 = p^2 - 2pp_z\cos\varphi + p_z^2. \tag{4.64}$$

4. Energy conservation reads:

$$\sqrt{p^2c^2 + m^2c^4} = \sqrt{p_z^2c^2 + m^2c^4} + h\nu, \tag{4.65}$$

or equivalently:

$$\frac{1}{\sqrt{1-\beta^2}}mc^2 = \frac{1}{\sqrt{1-\beta_f^2}}mc^2 + h\nu. \tag{4.66}$$

5. Squaring both sides, we obtain:

$$p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2}, \quad \text{where } E \text{ denotes the initial energy of the electron.} \tag{4.67}$$

6. Comparing the two expressions for p_z^2, we can write:

$$p^2 - 2pp_z\cos\varphi + p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2}, \tag{4.68}$$

The differential cross-section can then be rewritten as:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{a_{\min}^2}{16}\cdot\frac{1}{\sin^4(\theta/2)}}. \tag{4.57}$$

from which, after simplification, we get:

$$\cos\varphi = \frac{h\nu}{pc}\left(1 - \frac{E}{pc}\right) + \frac{h\nu}{2pc}, \tag{4.69}$$

with $E = \gamma mc^2$, $p = \gamma mv$, $p_z = \frac{nh\nu}{c}$, so that:

$$\cos\theta = \frac{1}{n\beta}\left(1 - \frac{1}{2}\frac{1}{\gamma^2}\right). \tag{4.70}$$

7. Finally:

$$\cos\theta = \frac{1}{n\beta}\left[1 + (n^2-1)\frac{1}{2\gamma^2}\right]. \tag{4.71}$$

Since $E = \gamma mc^2$, this can also be written as:

$$\cos\theta = \frac{1}{n\beta}\left[1 + \frac{n^2-1}{2}\left(1-\beta^2\right)\right]. \tag{4.72}$$

8. We must have:

$$\frac{1}{n\beta}\left[1 + (n^2-1)\frac{1}{2\gamma^2}\right] \leq 1. \tag{4.73}$$

Since the bracketed term is clearly greater than 1, it is necessary (though not sufficient) that:

$$\beta > \frac{1}{n}. \tag{4.74}$$

9. Photons are emitted between $v = 0$ and a frequency ν_{max} such that $\cos\theta = 1$, i.e.:

$$0 \leq \nu \leq \frac{E}{h}\left(1 - \frac{1}{n\beta}\right), \quad \text{with } E = \nu_{\text{max}} h. \tag{4.75}$$

10. The most energetic photons are emitted in the direction $\theta = 0$.

11. All photons are emitted within a cone of half-angle φ, corresponding to the angle θ for a photon of zero frequency:

$$\varphi = \arccos\left(\frac{1}{n\beta}\right) = \arccos\left(\frac{1}{n}\right) \simeq 20°. \tag{4.76}$$

12. For the effect to occur, one needs $\nu > \frac{1}{n}$, i.e. $\beta > \frac{1}{n}$, hence:

$$E > \frac{1}{\sqrt{1 - \frac{1}{n^2}}} mc^2. \tag{4.77}$$

For an electron, this means $E > 0.77$ MeV, and for a proton, $E > 1.4$ GeV.

4.4 Pulsed Magnetic Field Machine

1. **Magnetic field of the coil**

   (a) For a circular loop of radius R, Biot-Savart's law gives the field along the z-axis:

   $$B_z(z,t) = \frac{\mu_0 I(t) R^2}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.78}$$

   This is obtained by integrating over the loop, exploiting its circular symmetry.

   (b) For $z \gg R$, we can approximate $\left(z^2+R^2\right)^{3/2} \simeq z^3$. Thus,

   $$B_z(z,t) \sim \frac{\mu_0 I(t) R^2}{2z^3}, \tag{4.79}$$

   which is the expression for the field of a magnetic dipole with moment $m = I(t)R^2$.

2. **Induced electric field in biological tissue**

   (a) Faraday's local law in cylindrical coordinates reads (assuming the induced electric field is purely azimuthal):

   $$(\nabla \times \mathbf{E})_z = \frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\partial B_z}{\partial t}. \tag{4.80}$$

   Differentiating B_z with respect to time:

   $$\frac{\partial B_z}{\partial t} = \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\dot{I}(t). \tag{4.81}$$

   The local equation thus becomes:

   $$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.82}$$

   (b) **Integration for $r < R$:** Integrating from 0 to r, imposing $E_\theta(0,t) = 0$ (to avoid a singularity):

   $$\int_0^r \frac{\partial(r'E_\theta(r',t))}{\partial r'}\frac{\mathrm{d}r'}{r'} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\int_0^r \mathrm{d}r'. \tag{4.83}$$

   The solution obtained is:

   $$rE_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\cdot\frac{r^2}{2}, \tag{4.84}$$

   which leads to:

   $$E_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}\,r \quad \text{for } r \le R. \tag{4.85}$$

   Integration for $r > R$: For $r > R$, since the magnetic flux remains confined within the coil's area, it is more appropriate to use Faraday's integral law. Considering a circular loop of radius $r > R$, Faraday's law gives:

   $$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = 2\pi r\,E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{4.86}$$

   where the flux Φ is that through the coil area, i.e.:

   $$\Phi = \pi R^2\,B_z(z,t) = \pi R^2\frac{\mu_0 I(t) R^2}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.87}$$

The time derivative of Φ is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \pi R^2 \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}} \dot{I}(t). \tag{4.88}$$

Thus,

$$2\pi r\, E_\theta = -\pi \frac{\mu_0 R^4\, \dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}, \tag{4.89}$$

and therefore for $r > R$:

$$E_\theta(r,t) = -\frac{\mu_0 R^4\, \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}. \tag{4.90}$$

Summary:

$$E_\theta(r,t) = \begin{cases} -\dfrac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}\, r, & r \le R, \\ -\dfrac{\mu_0 R^4\, \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}, & r \ge R. \end{cases} \tag{4.91}$$

Continuity check: At $r = R$, the inner solution gives:

$$E_\theta(R,t) = -\frac{\mu_0 R^3\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}, \tag{4.92}$$

and the outer solution gives exactly the same result. Continuity is thus ensured.

3. **Effect on motor neurons**

   (a) The induced voltage across a disk of radius a is given by:

   $$V = \int_0^a E(r,t)\, dr. \tag{4.93}$$

   Using the expression of $E_\theta(r,t)$ for $r \le R$ (assuming $a \le R$ for simplicity), we have:

   $$V = -\frac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \int_0^a r\, dr = -\frac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \cdot \frac{a^2}{2}. \tag{4.94}$$

   Thus,

   $$V = -\frac{\mu_0 R^2\, a^2\, \dot{I}(t)}{8\left(z^2+R^2\right)^{3/2}}. \tag{4.95}$$

   (b) To activate the neuron, we need $|V| \ge V_{\text{threshold}}$. Therefore the activation condition is:

   $$\frac{\mu_0 R^2\, a^2\, |\dot{I}(t)|}{8\left(z^2+R^2\right)^{3/2}} \ge V_{\text{threshold}}. \tag{4.96}$$

4. **Effect of pulsed magnetic fields on muscles** When the magnetic stimulation machine delivers rapid pulses, the time variation of the magnetic field induces an electric field in surrounding tissues. In muscles, this electric field can depolarise cell membranes by activating ion channels, generating an action potential. This excitation leads to involuntary muscle contraction, used in physiotherapy to enhance muscle rehabilitation, increase blood circulation, and reduce pain.

4.5 Metric of a Sphere

1. Using that $\mathrm{d}(\cos u) = -\sin u\,\mathrm{d}u$ and $\mathrm{d}(\sin u) = \cos u\,\mathrm{d}u$, we get

$$\frac{\mathrm{d}x^2}{R^2} = [-\sin\theta\sin\varphi\mathrm{d}\varphi + \cos\theta\cos\varphi\mathrm{d}\theta]^2 \tag{4.97}$$

$$= (\sin\theta\sin\varphi\mathrm{d}\varphi)^2 - 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\cos\varphi\mathrm{d}\theta)^2, \tag{4.98}$$

$$\frac{\mathrm{d}y^2}{R^2} = [\sin\theta\cos\varphi\mathrm{d}\varphi + \cos\theta\sin\varphi\mathrm{d}\theta]^2 \tag{4.99}$$

$$= (\sin\theta\cos\varphi\mathrm{d}\varphi)^2 + 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\sin\varphi\mathrm{d}\theta)^2, \tag{4.100}$$

$$\frac{\mathrm{d}z^2}{R^2} = \sin^2\theta\mathrm{d}\theta^2. \tag{4.101}$$

Adding these terms and using $\cos^2 + \sin^2 = 1$, we obtain

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{4.102}$$

2. From equation (4.102), factoring by $\mathrm{d}\theta^2$ inside the square root, we have

$$\mathrm{d}s = R\sqrt{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2} \tag{4.103}$$

$$= R\sqrt{1 + \sin^2\theta\,\varphi'^2}\mathrm{d}\theta, \quad \varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}\theta} \tag{4.104}$$

$$= R\mathcal{L}\mathrm{d}\theta. \tag{4.105}$$

We notice that $\partial_\varphi\mathcal{L} = 0$, so φ is a cyclic variable. Thus,

$$\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R} \tag{4.106}$$

where λ is a constant.

3.

$$\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R} \tag{4.107}$$

$$\implies \frac{\varphi'\sin^2\theta}{\sqrt{1+\sin^2\theta\varphi'^2}} = \lambda \tag{4.108}$$

$$\implies \varphi'^2(\sin^4\theta - \lambda^2\sin^2\theta) = \lambda^2 \tag{4.109}$$

$$\implies \mathrm{d}\varphi = \lambda\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{1-\frac{\lambda^2}{\sin^2\theta}}}. \tag{4.110}$$

Integrating,

$$\varphi - \varphi_0 = \lambda\int^\theta \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1-\frac{\lambda^2}{\sin^2\alpha}}} \tag{4.111}$$

$$=_{u=\cot\alpha} -\lambda\int^{\cot\theta}\frac{\mathrm{d}u}{\sqrt{1-\lambda^2(1+u^2)}} \tag{4.112}$$

$$=_{t=\frac{u}{\beta}} -\frac{\lambda}{\beta}\int^{\frac{\cot\theta}{\beta}}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad \beta^2 = 1-\lambda^2 \tag{4.113}$$

$$= \arccos\left(\frac{\cot\theta}{\beta}\right). \tag{4.114}$$

Thus,

$$\beta \cos(\varphi - \varphi_0) = \cot\theta, \tag{4.115}$$

$$\beta \sin\theta \cos(\varphi - \varphi_0) = \cos\theta. \tag{4.116}$$

Using some trigonometric formulas, we obtain

$$R \times \big(\beta \cos\varphi_0 \cos\varphi \sin\theta + \beta \cos\varphi_0 \sin\varphi \sin\theta\big) = \cos\theta \tag{4.117}$$

$$\implies ax + by - z = 0, \tag{4.118}$$

where we substituted using spherical coordinates, with $a = \beta \cos\varphi_0 = b$.

4.6 Blackbody Radiation

4.6.1 Number of Modes Excited per Frequency Unit

1. This is the D'Alembert equation in vacuum,

$$\Box \mathbf{E} = 0. \tag{4.119}$$

2. The cavity enforces a stationary solution, thus

$$\mathbf{E} = \cos \omega t \sum_{\mu=1}^{3} E^{\mu} \sin(k_{\mu} x^{\mu}) \mathbf{e}_{\mu}. \tag{4.120}$$

For each μ, the boundary condition is $\mathbf{E}(x^{\mu} = L) = \mathbf{0}$. Hence,

$$\sin(k_{\mu} L) = 0, \tag{4.121}$$

$$k_{\mu} L = n_{\mu} \pi, \tag{4.122}$$

$$k_{\mu} = \frac{n_{\mu} \pi}{L}. \tag{4.123}$$

3. We know that the norm of $\mathbf{k}$ equals the sum over each component,

$$\|\mathbf{k}\|^2 = \sum_{\mu} \left(\frac{n_{\mu} \pi}{L} \right)^2, \tag{4.124}$$

$$\left(\frac{2\pi}{\lambda} \right)^2 = \frac{\pi^2}{L^2} \sum_{\mu} n_{\mu}^2, \tag{4.125}$$

$$r^2 = \left(\frac{2L}{\lambda} \right)^2 = \sum_{\mu} n_{\mu}^2. \tag{4.126}$$

4. The volume of modes up to frequency $\|\mathbf{k}\|$ is

$$V(\|\mathbf{k}\|) = \frac{4}{3} \pi \|\mathbf{k}\|^3. \tag{4.127}$$

The number of modes is the mode volume divided by the volume of a single mode, with some factors. Since $k_{\mu} = \frac{\pi}{L} n_{\mu}$ and $n_{\mu} \in \mathbb{N}^*$ (factor $\times \frac{1}{8}$), and polarization (factor $\times 2$), we have

$$\implies N = \frac{1}{8} \times 2 \times \frac{V(\|\mathbf{k}\|)}{\left(\frac{\pi}{L}\right)^3} = 2 \times \frac{4}{3} \pi r^3 \tag{4.128}$$

$$= \frac{1}{8} \times 2 \times \frac{\frac{4}{3} \pi \|\mathbf{k}\|^3}{\pi^3} L^3 \tag{4.129}$$

$$= \frac{1}{8} \times 2 \times \frac{4}{3} \pi \frac{\left(\frac{2\pi}{\lambda}\right)^3}{\pi^3} L^3 \tag{4.130}$$

$$= \pi \frac{8L^3}{3\lambda^3} \tag{4.131}$$

$$= \frac{8\pi \nu^3}{3c^3} L^3, \tag{4.132}$$

$$\implies \frac{\mathrm{d}N}{\mathrm{d}\nu} = \frac{8\pi \nu^2}{c^3} \mathcal{V} \tag{4.133}$$

4.6.2 Ultraviolet Catastrophe

1. The system is in contact with a thermostat at temperature T, and the system is closed.
2. In 1D,
$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2 \tag{4.134}$$
3.
$$p(W = \varepsilon) = \frac{1}{Z}\exp(-\beta\varepsilon) \tag{4.135}$$
We also have in 1D,
$$Z = \frac{1}{h}\int_{\mathbb{R}^2} e^{-\beta\mathcal{H}}\mathrm{d}q\mathrm{d}p \tag{4.136}$$
Hence,
$$\int_{\mathbb{R}} e^{-\beta\frac{p^2}{2m}}dp = \sqrt{\frac{2m\pi}{\beta}} \tag{4.137}$$
And,
$$\int_{\mathbb{R}} e^{-\beta\frac{m\omega^2 q^2}{2}}dq = \sqrt{\frac{2\pi}{m\omega^2\beta}} \tag{4.138}$$
Hence,
$$Z = \frac{1}{h}\frac{2\pi}{\omega\beta} = \frac{1}{h}\frac{T}{\beta} \tag{4.139}$$
4. We use the formula for the average energy,
$$\langle W\rangle = -\partial_\beta \ln Z = \partial_\beta \ln\beta = \frac{1}{\beta} = k_B T \tag{4.140}$$
5. It is then obvious to say that thanks to eq. 4.144 and the previous question,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}k_B T \tag{4.141}$$
Hence $u \propto \nu^2$, which implies, $\int_{\mathbb{R}^+} u\mathrm{d}\nu \propto \int_{\mathbb{R}^+}\nu^2\mathrm{d}\nu$, which diverges.

4.6.3 Planck's Law

1. The energy levels are discrete, so we sum:
$$Z = \sum_n e^{-\beta W_n} = \frac{1}{1 - e^{-\beta W_1}} \tag{4.142}$$
Thus, the average energy becomes by the same calculation,
$$-\partial_\beta \ln Z = \frac{h\nu}{e^{\beta h\nu} - 1} \tag{4.143}$$
Using, $W_1 = h\nu$.
2. It is then obvious that,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu} - 1} \tag{4.144}$$

4.6.4 Energy flux emitted by a black body

1. Monochromatic energy flux in a given direction.

   The directional spectral intensity $I_\nu(\theta,\varphi)$ is defined as the energy transported per unit area, time, frequency, and steradian, in direction (θ,φ).

   The monochromatic energy flux emitted in direction (θ,φ) relative to the surface normal is:

$$\mathrm{d}\Phi_\nu = I_\nu(\theta,\varphi)\cos\theta\,\mathrm{d}\Omega, \tag{4.145}$$

   where $\mathrm{d}\Omega$ is the solid angle element around this direction, and $\cos\theta$ comes from the projection of the flux on the normal to the surface (cf. fig 4.1).

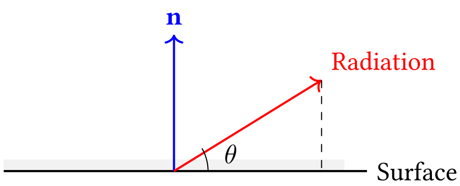

Figure 4.1: The radiation is emitted with an angle θ relative to the normal: only $\cos\theta$ contributes to the flux through the surface. Indeed, it goes out in all directions, so we integrate over $[0,\frac{\pi}{2}]$, and only the contribution of $\cos\theta$ (the projection) matters.

2. Total energy flux emitted at frequency ν.

   The total energy flux $I(\nu)$ emitted at frequency ν per unit surface is obtained by integrating the elementary flux over the entire outgoing hemisphere (i.e. directions such that $0 \le \theta \le \pi/2$, cf. fig 4.2):

$$I(\nu) = \int_{\Omega_+} I_\nu(\theta,\varphi)\cos\theta\,\mathrm{d}\Omega. \tag{4.146}$$

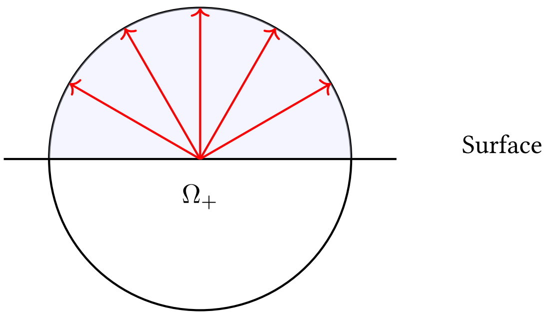

Figure 4.2: The radiation goes out in all directions of the hemisphere Ω_+: we integrate only for $\theta \in [0,\pi/2]$.

3. Case of isotropic radiation

   If the radiation is isotropic, we have $I_\nu(\theta,\varphi) = I_\nu$ = constant (independent of direction). We can then take I_ν out of the integral:

$$I(\nu) = I_\nu \int_{\Omega_+} \cos\theta\,\mathrm{d}\Omega. \tag{4.147}$$

But:

$$\int_{\Omega_+} \cos\theta \, \mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi. \tag{4.148}$$

Calculating,

$$\int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta = \frac{1}{2}, \quad \text{and} \quad \int_0^{2\pi} \mathrm{d}\varphi = 2\pi. \tag{4.149}$$

Hence,

$$I(\nu) = I_\nu \cdot 2\pi \cdot \frac{1}{2} = \pi I_\nu. \tag{4.150}$$

4. Total emitted intensity (all frequencies combined)

We want to show that the spectral volumetric energy density $u(\nu)$ can be expressed as a function of the directional intensity $I_\nu(\mathbf{n})$ by:

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \, \mathrm{d}\Omega. \tag{4.151}$$

- $u(\nu)\,\mathrm{d}\nu$ represents the electromagnetic energy contained in a unit volume, for waves whose frequency is between ν and $\nu + \mathrm{d}\nu$.
- $I_\nu(\mathbf{n})$ is the spectral intensity in the direction $\mathbf{n}$, that is, the energy transported per unit time, per unit perpendicular surface, per unit frequency, per unit solid angle.

Consider an elementary surface $\mathrm{d}s$ and a radiation beam incident along a direction $\mathbf{n}$ making an angle θ with the normal to $\mathrm{d}s$.

The volume V swept by the rays in the direction $\mathbf{n}$ during a short time interval $\mathrm{d}t$ is given by:

$$\mathrm{d}V = c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta. \tag{4.152}$$

The energy transported through the surface $\mathrm{d}s$ by these rays during this time is:

$$\mathrm{d}E = I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega. \tag{4.153}$$

We deduce that the energy per unit volume associated with the direction $\mathbf{n}$ is:

$$\frac{\mathrm{d}E}{\mathrm{d}V} = \frac{I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega}{c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta} = \frac{I_\nu(\mathbf{n})}{c}\mathrm{d}\Omega. \tag{4.154}$$

To obtain the total energy density, we sum over all propagation directions on the unit sphere:

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \, \mathrm{d}\Omega. \tag{4.155}$$

If the radiation is isotropic, then $I_\nu(\mathbf{n}) = I_\nu$ is independent of direction. The integral becomes:

$$u(\nu) = \frac{I_\nu}{c} \int_{S^2} \mathrm{d}\Omega = \frac{I_\nu}{c} \cdot 4\pi. \tag{4.156}$$

Hence,

$$\boxed{u(\nu) = \frac{4\pi}{c} I_\nu} \tag{4.157}$$

5. Relation between total intensity and $u(\nu)$

   We take again the previous expression:

$$I = \int_0^\infty \pi I_\nu \, \mathrm{d}\nu, \tag{4.158}$$

and substitute $I_\nu = \frac{c}{4\pi} u(\nu)$:

$$I = \int_0^\infty \pi \cdot \frac{c}{4\pi} u(\nu) \, \mathrm{d}\nu = \frac{c}{4} \int_0^\infty u(\nu) \, \mathrm{d}\nu. \tag{4.159}$$

4.6.5 Stefan's Law

1. We previously demonstrated that,

$$I(T) = \frac{c}{4} \int_{\mathbb{R}^+} u(\nu, T) \mathrm{d}\nu \tag{4.160}$$

Replacing with what was obtained in eq. 4.144,

$$I = \frac{c}{4} \frac{8\pi}{c^3} \int_{\mathbb{R}^+} \frac{h\nu^3}{e^{\beta h\nu} - 1} \mathrm{d}\nu \tag{4.161}$$

$$\underset{x=\beta h\nu}{=} \frac{2\pi k_B^4}{h^3 c^2} T^4 \int_0^\infty \frac{x^3}{e^x - 1} \mathrm{d}x \tag{4.162}$$

Recall that $\beta = \frac{1}{k_B T}$.

2. By performing a series expansion, one easily eliminates division by zero. Indeed, near zero,

$$e^x - 1 \underset{0}{=} x + o(x) \implies \frac{x^3}{e^x - 1} \underset{0}{=} x^2 + o(x^2) \tag{4.163}$$

which converges well at zero. At infinity, the exponential ensures convergence of the integral.

$$\int_{\mathbb{R}^+} \frac{x^3}{e^x - 1} \mathrm{d}x = \int_{\mathbb{R}^+} \mathrm{d}x \, x^3 e^{-x} \frac{1}{1 - e^{-x}} \tag{4.164}$$

$$=_{\text{DSE}} \int_{\mathbb{R}^+} \mathrm{d}x \, x^3 \sum_{n \in \mathbb{N}^*} e^{-nx} \tag{4.165}$$

$$\underset{u=nx}{=} \sum_{n \in \mathbb{N}^*} \frac{1}{n^4} \int_{\mathbb{R}^+} \mathrm{d}u \, u^3 e^{-u} \tag{4.166}$$

$$= \zeta(4)\Gamma(4) \tag{4.167}$$

$$= 6\zeta(4) \tag{4.168}$$

3. Thanks to Fourier theory, one can show that $\zeta(4) = \frac{\pi^4}{90}$. We then have,

$$I(T) = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4, \tag{4.169}$$

4.6.6 Application: Solar mass loss due to electromagnetic radiation

We consider the Sun as a black body at temperature $T = 5775\,\text{K}$. The total power radiated by the Sun is given by Stefan-Boltzmann law:

$$P = I \cdot S = \sigma T^4 \cdot 4\pi R^2, \tag{4.170}$$

where

$$\sigma = 5{,}67 \times 10^{-8}\ \text{W.m}^{-2}\text{K}^{-4}, \quad R = 6{,}96 \times 10^{8}\ \text{m} \tag{4.171}$$

is the radius of the Sun.

Let's calculate P:

$$P = 5{,}67 \times 10^{-8} \times (5775)^4 \times 4\pi (6{,}96 \times 10^8)^2. \tag{4.172}$$

We estimate:

$$(5775)^4 \simeq 1{,}11 \times 10^{15}, \tag{4.173}$$

$$4\pi(6{,}96 \times 10^8)^2 = 4\pi \times 4{,}84 \times 10^{17} \simeq 6{,}08 \times 10^{18}. \tag{4.174}$$

Thus,

$$P \simeq 5{,}67 \times 10^{-8} \times 1{,}11 \times 10^{15} \times 6{,}08 \times 10^{18} \simeq 3{,}83 \times 10^{26}\ \text{W}. \tag{4.175}$$

According to Einstein's mass-energy equivalence relation,

$$E = mc^2, \tag{4.176}$$

the mass loss rate $\dot{m}$ per unit time related to this radiated power is

$$\dot{m} = \frac{P}{c^2}, \tag{4.177}$$

with $c = 3{,}00 \times 10^8$ m/s.

Hence,

$$\dot{m} = \frac{3{,}83 \times 10^{26}}{(3{,}00 \times 10^8)^2} = \frac{3{,}83 \times 10^{26}}{9 \times 10^{16}} \simeq 4{,}26 \times 10^9\ \text{kg/s}. \tag{4.178}$$

Knowing that the age of the Sun is about $t = 4{,}6 \times 10^9$ years, i.e.

$$t = 4{,}6 \times 10^9 \times 3{,}15 \times 10^7 \simeq 1{,}45 \times 10^{17}\ \text{s}, \tag{4.179}$$

the total lost mass is

$$\Delta m = \dot{m} \times t = 4{,}26 \times 10^9 \times 1{,}45 \times 10^{17} \simeq 6{,}18 \times 10^{26}\ \text{kg}. \tag{4.180}$$

In number of Earth masses, with $m_T = 6 \times 10^{24}$ kg,

$$\frac{\Delta m}{m_T} = \frac{6{,}18 \times 10^{26}}{6 \times 10^{24}} \simeq 103. \tag{4.181}$$

Thus, the Sun loses about $4{,}3 \times 10^9$ kg/s by radiation. Since its formation, it has lost about 100 times the mass of the Earth.

4.7 Minimization of the gravitational potential by a ball

4.7.1 Hadamard's formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and let Ω_ε be a smooth deformation of Ω such that, for $x \in \partial\Omega$,

$$x \;\mapsto\; x + \varepsilon\, f(x)\, n(x), \tag{4.182}$$

extended on all Ω. We want to prove:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\,\mathrm{d}^3x \;=\; \int_{\partial\Omega} F(x)\, f(x)\,\mathrm{d}S(x), \tag{4.183}$$

where $\mathrm{d}S$ is the surface element on $\partial\Omega$.

1. **Study of the function** $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$**.**

   (a) *Differentiability of* $\det$.
   Recall that for $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{R})$,

   $$\det(M) \;=\; \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}. \tag{4.184}$$

   It is thus a polynomial in the n^2 variables m_{ij}. Any polynomial function $\mathbb{R}^{n^2} \to \mathbb{R}$ is of class $\mathcal{C}^\infty$. In particular, $\det$ is differentiable at every point of $\mathcal{M}_n(\mathbb{R})$, notably near the identity I.

   (b) *Expansion of* $\det(I + \varepsilon M)$.
   We want to show:

   $$\forall M \in \mathcal{M}_n(\mathbb{R}), \quad \det\bigl(I + \varepsilon M\bigr) \underset{\varepsilon\to 0}{=} 1 \;+\; \varepsilon\ \mathrm{Tr}(M) \;+\; o(\varepsilon), \tag{4.185}$$

   which implies $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \det(I + \varepsilon M) = \mathrm{Tr}(M)$.
   It suffices to write M in upper triangular form, then the determinant is the product of the eigenvalues!
   Thus,

   $$\det(I + \varepsilon M) = \prod_{i=1}^{n}(1 + \varepsilon\lambda_i) = 1 + \varepsilon \sum_{i=1}^{n} \lambda_i + O(\varepsilon^2) = 1 + \varepsilon\,\mathrm{Tr}\,M + o(\varepsilon) \tag{4.186}$$

   which concludes the proof.

   (c) We reduce to the previous case by factoring out X.

   $$\begin{aligned} \det(X + H) &= \det X \det\bigl(I + X^{-1}H\bigr) && (4.187)\\ &= \det X\Bigl(1 + \mathrm{tr}\bigl(X^{-1}H\bigr) + o(\|H\|)\Bigr) && (4.188)\\ &= \det X + \mathrm{tr}\bigl({}^t\mathrm{Com}(X)H\bigr) + o(\|H\|) && (4.189)\end{aligned}$$

   Thus we have,

   $$\mathrm{d}(\det(H))\,(X) = \mathrm{Tr}\bigl({}^t\mathrm{Com}(X)H\bigr) \tag{4.190}$$

2. **Change of variables and calculation of the Jacobian.**
   We perform the change of variable

$$x = x(u) = u + \varepsilon\, f(u)\, n(u), \qquad u \in \Omega. \tag{4.191}$$

To compute $\det\left(\frac{\partial x}{\partial u}\right)$ at first order in ε, we write

$$x_i(u) = u_i + \varepsilon\, f(u)\, n_i(u), \quad i = 1, \dots, n. \tag{4.192}$$

Then

$$\frac{\partial x_i}{\partial u_j} = \delta_{ij} + \varepsilon \left(\partial_j f(u)\right) n_i(u) + \varepsilon\, f(u)\, \partial_j n_i(u). \tag{4.193}$$

Let the matrix $A(u) = \left(\partial_j f\, n_i + f\, \partial_j n_i\right)_{i,j}$. We have $\dfrac{\partial x}{\partial u} = I + \varepsilon\, A(u)$. By the previous expansion,

$$\det\left(\frac{\partial x}{\partial u}\right) = \det\left(I + \varepsilon\, A(u)\right) = 1 + \varepsilon\ \mathrm{Tr}\left(A(u)\right) + o(\varepsilon). \tag{4.194}$$

Noticing that $\mathrm{Tr}(A(u)) = \nabla\cdot\left(f\, n\right)$, we obtain

$$\det\left(\frac{\partial x}{\partial u}\right) = 1 + \varepsilon\, \nabla\cdot\left(f\, n\right)(u) + o(\varepsilon). \tag{4.195}$$

3. **Expansion of $F(x + \varepsilon\, v(x))$.**
   Let $F : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$. Fixing x, define $\varphi(\varepsilon) = F\left(x + \varepsilon\, v(x)\right)$. By the chain rule in dimension 1,

$$\varphi'(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(x + \varepsilon v(x)) = v(x)\ \cdot \nabla F\left(x + \varepsilon\, v(x)\right). \tag{4.196}$$

In particular, for $\varepsilon \to 0$,

$$\varphi(\varepsilon) = \varphi(0) + \varepsilon\, \varphi'(0) + o(\varepsilon) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon). \tag{4.197}$$

Hence

$$\forall x \in \mathbb{R}^n, \quad F\left(x + \varepsilon\, v(x)\right) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon). \tag{4.198}$$

4. **Derivation of Hadamard's formula.**
   We perform the change $x(u)$ in $\displaystyle\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x$. Then

$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F\left(x(u)\right)\ \det\left(\frac{\partial x}{\partial u}\right) \mathrm{d}^3 u. \tag{4.199}$$

From the two previous points,

$$F\left(x(u)\right) = F(u) + \varepsilon\, f(u)\, n(u) \cdot \nabla F(u) + o(\varepsilon), \quad \det\left(\tfrac{\partial x}{\partial u}\right) = 1 + \varepsilon\, \nabla\cdot\left(f\, n\right)(u) + o(\varepsilon). \tag{4.200}$$

Multiplying,

$$F\left(x(u)\right) \det\left(\frac{\partial x}{\partial u}\right) = F(u) + \varepsilon \left[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\right](u) + o(\varepsilon). \tag{4.201}$$

Therefore,

$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F(u)\, \mathrm{d}^3 u + \varepsilon \int_{\Omega} \left[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\right](u)\, \mathrm{d}^3 u + o(\varepsilon). \tag{4.202}$$

Then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\,\mathrm{d}^3x = \int_\Omega \nabla\cdot\big(F\,f\,n\big)(u)\,\mathrm{d}^3u, \tag{4.203}$$

using the product rule. Finally, by the divergence theorem,

$$\int_\Omega \nabla\cdot\big(F\,f\,n\big)\,\mathrm{d}^3u = \int_{\partial\Omega} F\,f\,n\cdot n\,\mathrm{d}S(u) = \int_{\partial\Omega} F\,f\,\mathrm{d}S. \tag{4.204}$$

This concludes the proof of Hadamard's formula (3.48).

4.7.2 Connection with the gravitational potential

1. **Sign of $E[\Omega]$ and definition of $\mathcal{I}[\Omega]$.**
   We have

$$E[\Omega] = -\frac{G}{2}\rho^2 \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'. \tag{4.205}$$

   Since $G > 0$ and $\rho > 0$, it immediately follows that $E[\Omega] < 0$. Minimizing $E[\Omega]$ is therefore equivalent to *maximizing*

$$\mathcal{I}[\Omega] := \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'. \tag{4.206}$$

2. **Calculation of the potential at the center of a ball.**
   Suppose $\Omega = B(0,R)$ with fixed volume $\frac{4}{3}\pi R^3 = V$. The density is ρ. For $x = 0$,

$$U(0) = -G\,\rho \int_\Omega \frac{1}{|x'|}\,\mathrm{d}^3x' = -G\,\rho \int_0^R \int_{S^2} \frac{1}{r}\,r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r. \tag{4.207}$$

   In spherical coordinates,

$$\int_{S^2} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi, \quad \text{and} \quad \int_0^R \frac{r^2}{r}\,\mathrm{d}r = \int_0^R r\,\mathrm{d}r = \frac{R^2}{2}. \tag{4.208}$$

   Thus

$$U(0) = -G\,\rho\cdot 4\pi\cdot\frac{R^2}{2} = -2\pi\,G\,\rho\,R^2. \tag{4.209}$$

   Hence the explicit expression of the potential at the center.

4.7.3 The sphere?

1. **First variation of $\mathcal{F}$.**
   We write $\mathcal{F}[\Omega_\varepsilon]$ and apply Hadamard's formula with $F(x) = \int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'$. Then

$$\delta\mathcal{F} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{1}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.210}$$

   Thus, using Hadamard's formula for Ω^2,

$$\delta\mathcal{F} = 2\int_{\partial\Omega}\left(\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x). \tag{4.211}$$

2. **Introduction of the Lagrange multiplier λ.**
We want to minimize $\mathcal{F}$ under the constraint $V[\Omega] = V$. We define the Lagrangian functional

$$\mathcal{L}[\Omega] := \mathcal{F}[\Omega] - \lambda V[\Omega], \quad \lambda \in \mathbb{R}. \tag{4.212}$$

Its first variation writes

$$\delta\mathcal{L} = \delta\mathcal{F} - \lambda\,\delta V = 2\int_{\partial\Omega}\left(\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x) \;-\; \lambda\int_{\partial\Omega} f(x)\,\mathrm{d}S(x). \tag{4.213}$$

By linearity,

$$\delta\mathcal{L} = \int_{\partial\Omega}\left(2\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda\right) f(x)\,\mathrm{d}S(x). \tag{4.214}$$

3. **Stationary condition for the ball.**
For $\delta\mathcal{L} = 0$ for *all* perturbations f, it is necessary and sufficient that

$$2\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda \;=\; 0, \quad \text{for all } x \in \partial\Omega. \tag{4.215}$$

In other words, the function $x \mapsto \displaystyle\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x'$ is constant on $\partial\Omega$.
If $\Omega = B(0,R)$ is a ball, then by spherical symmetry, for every $x \in \partial B(0,R)$ (i.e. $|x| = R$), the integral $\displaystyle\int_{B(0,R)}\frac{1}{|x-x'|}\,\mathrm{d}^3x'$ depends only on $|x| = R$.
Thus it is *constant* on ∂B. We deduce that the ball satisfies the stationary condition $\delta\mathcal{L} = 0$ for all f.

4. **(*Bonus*) Second variation and local minimum.**
To show that the ball is a *local minimum* of $\mathcal{F}$ under volume constraint V, one must verify that the second variation $\delta^2\mathcal{L}[f]$ is strictly positive for any perturbation $f \neq 0$ satisfying $\int_{\partial\Omega} f\,\mathrm{d}S = 0$.
Without full details here, the second variation can be written as a bilinear form:

$$\delta^2\mathcal{F}[f] = \int_{(\partial\Omega)^2} K(x,x')\,f(x)\,f(x')\,\mathrm{d}S(x)\,\mathrm{d}S(x') \;+\; \int_{\partial\Omega} f(x)^2\kappa(x)\mathrm{d}S(x), \tag{4.216}$$

with kernel $K(x,x') = \frac{1}{|x-x'|}$ and $\kappa(x)$ the mean curvature at x.
For the ball, thanks to the spherical harmonics expansion, one shows this form is strictly positive on $\{\, f \mid \int_{\partial\Omega} f\,\mathrm{d}S = 0 \,\}$. This proves the ball is a local minimum.

5. **Physical conclusion.**
The ball minimizes the internal gravitational energy for a fixed volume. In physics, this explains that in the approximation of a massive self-gravitating body at rest, the stationary configuration of least energy is spherical. This is why large objects in the Universe (stars, planets in the absence of tidal forces or rapid rotation) tend to a spherical shape.

4.8 Relativistic motion of a charged particle

4.8.1 Relativistic Lagrangian of a Charged Particle in an Electromagnetic Field

1. **Free particle and relativistic action.**
   The principle of least action requires that the action be a Lorentz scalar. The simplest scalar is the spacetime interval $\mathrm{d}s$, defined by:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{4.217}$$

   The action for a free particle of mass m is thus:

$$S = -mc\int \mathrm{d}s = -mc^2\int \sqrt{1-\frac{\mathbf{v}^2}{c^2}}\,\mathrm{d}t. \tag{4.218}$$

   The associated Lagrangian is:

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}. \tag{4.219}$$

2. **Interaction with an electromagnetic field.**
   We introduce the four-potential $A^\mu = (\phi/c, \mathbf{A})$. We seek an interaction term of the scalar form $L_{\text{int}} = qA_\mu \dot{x}^\mu$. In standard coordinates:

$$\mathcal{L}_{\text{int}} = q\mathbf{A}\cdot\mathbf{v} - q\phi. \tag{4.220}$$

   The total Lagrangian is therefore:

$$\mathcal{L}_{\text{tot}} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{v} - q\phi. \tag{4.221}$$

3. **Generalized momentum.**
   The generalized momentum is defined as:

$$\mathbf{p} = \frac{\partial\mathcal{L}_{\text{tot}}}{\partial\mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} + q\mathbf{A} = \gamma m\mathbf{v} + q\mathbf{A}. \tag{4.222}$$

4. **Euler–Lagrange equations.**
   Applying the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right) = \frac{\partial\mathcal{L}}{\partial\mathbf{x}}. \tag{4.223}$$

   The left-hand side becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) + q\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}. \tag{4.224}$$

   The right-hand side yields:

$$q\nabla(\mathbf{A}\cdot\mathbf{v}) - q\nabla\phi. \tag{4.225}$$

   Using:

$$\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}), \tag{4.226}$$

   and:

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \partial_t\mathbf{A} + (\mathbf{v}\cdot\nabla)\mathbf{A}, \tag{4.227}$$

we find that the $(\mathbf{v}\cdot\nabla)\mathbf{A}$ terms cancel out, giving:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q\left[-\partial_t\mathbf{A} - \nabla\phi + \mathbf{v}\times(\nabla\times\mathbf{A})\right]. \tag{4.228}$$

Recognizing the electric and magnetic fields:

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{4.229}$$

we obtain the Lorentz force law:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{4.230}$$

5. **Covariant formulation.**
We parameterize the worldline using the proper time τ:

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + qA_\mu\dot{x}^\mu, \tag{4.231}$$

where the first term represents the free particle, and the second the interaction.

6. **Equation of motion.**
Applying the Euler–Lagrange equations in covariant form, we use the Lagrangian:

$$\mathcal{L} = -mc\sqrt{-\dot{x}^\mu\dot{x}_\mu} + qA_\mu\dot{x}^\mu, \tag{4.232}$$

with $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Since $\dot{x}^\mu\dot{x}_\mu = -c^2$, we get:

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = mc\frac{\dot{x}_\mu}{c} + qA_\mu. \tag{4.233}$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = m\ddot{x}_\mu + q\dot{x}^\nu\partial_\nu A_\mu, \tag{4.234}$$

$$\frac{\partial\mathcal{L}}{\partial x^\mu} = q\partial_\mu A_\nu\dot{x}^\nu, \tag{4.235}$$

yielding the equation of motion:

$$m\ddot{x}_\mu + q\dot{x}^\nu\partial_\nu A_\mu = q\dot{x}^\nu\partial_\mu A_\nu, \tag{4.236}$$

$$m\ddot{x}_\mu = q\dot{x}^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{4.237}$$

$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu, \tag{4.238}$$

where the antisymmetric electromagnetic tensor is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.239}$$

7. **Components of the tensor $F_{\mu\nu}$.**
$F_{\mu\nu}$ is antisymmetric and encodes the electric and magnetic fields. In Cartesian coordinates with $A^\mu = (\phi/c, \mathbf{A})$ and $x^\mu = (ct, \mathbf{x})$:

   - For $\mu = 0$, $\nu = i$:

$$F_{0i} = \frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{1}{c}\frac{\partial\phi}{\partial x^i} = -\frac{1}{c}E_i. \tag{4.240}$$

- For $\mu = i, \nu = j$:

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk} B_k. \tag{4.241}$$

Conclusion: the tensor $F_{\mu\nu}$ is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{4.242}$$

This explicitly shows how $F_{\mu\nu}$ encodes **E** and **B** in an inertial frame.

8. **Relativistic invariants.**
We compute:

$$I_1 = F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2}),$$
$$I_2 = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{8}{c}\mathbf{E}\cdot\mathbf{B}.$$

Characteristic cases:

- $\mathbf{E}^2 = c^2\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B} = 0$: electromagnetic plane wave.
- $I_1 > 0$: magnetic-field dominated; $I_1 < 0$: electric-field dominated.

9. **Gauge transformation.**
Under $A_\mu \to A_\mu + \partial_\mu \Lambda$, we have:

$$F_{\mu\nu} \to F_{\mu\nu}, \tag{4.243}$$

since mixed partial derivatives cancel. This leaves the equations of motion invariant: a **local gauge symmetry** associated with charge conservation via Noether's theorem.

4.8.2 Equations of Motion of a Charged Particle in a Plane Electromagnetic Wave – Solution

We consider a particle of mass m and charge q subjected to an electromagnetic field. Its motion is governed by the equation:

$$m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu, \tag{4.244}$$

where the dots denote derivatives with respect to the proper time τ, and we work in natural units: $c = 1$.

The potential is given by:

$$A^\mu(x) = a^\mu f(k_\nu x^\nu), \tag{4.245}$$

where a^μ is a constant four-vector, $f \in \mathcal{C}^1$, and k^μ is a lightlike four-vector satisfying $k^\mu k_\mu = 0$.

1. **Computation of the Electromagnetic Tensor.**

By definition:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.246}$$

We compute:

$$\partial^\mu A^\nu = a^\nu f'(k \cdot x) k^\mu, \qquad \partial^\nu A^\mu = a^\mu f'(k \cdot x) k^\nu, \tag{4.247}$$

so that:

$$F^{\mu\nu} = (k^\mu a^\nu - k^\nu a^\mu) f'(k \cdot x). \tag{4.248}$$

2. **Gauge Condition.**

   (a) We compute the Lorenz gauge condition:

   $$\partial_\mu A^\mu = a^\mu f'(k \cdot x) k_\mu. \tag{4.249}$$

   (b) Hence, the Lorenz condition $\partial_\mu A^\mu = 0$ implies:

   $$a^\mu k_\mu = 0, \tag{4.250}$$

   i.e., the polarization vector is orthogonal to the wave vector. This expresses the transversality of the electromagnetic wave.

3. **Equation of Motion.**

   (a) Using the expression of $F^{\mu\nu}$:

   $$F^{\mu\nu} \dot{x}_\nu = [k^\mu (a \cdot \dot{x}) - a^\mu (k \cdot \dot{x})] f'(k \cdot x). \tag{4.251}$$

   (b) The equation becomes:

   $$m \ddot{x}^\mu = q [k^\mu (a \cdot \dot{x}) - a^\mu (k \cdot \dot{x})] f'(k \cdot x). \tag{4.252}$$

4. **Integration of the Equation of Motion.**

   (a) Define $\phi(\tau) = k \cdot x(\tau)$, then:

   $$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = k \cdot \dot{x}, \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = k \cdot \ddot{x}. \tag{4.253}$$

   Using the motion equation and the identities $k^2 = 0$, $k \cdot a = 0$:

   $$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = 0 \quad \Rightarrow \quad \phi(\tau) = \omega\tau + \phi_0, \tag{4.254}$$

   with $\omega = k \cdot \dot{x} = \text{constant}$.

   (b) Let $u^\mu = \dot{x}^\mu$ and define $\alpha(\phi) = a \cdot u(\phi)$. Then:

   $$m\omega \frac{\mathrm{d}u^\mu}{\mathrm{d}\phi} = q [k^\mu \alpha(\phi) - a^\mu \omega] f'(\phi). \tag{4.255}$$

(c) Projecting on a_μ and using $a \cdot k = 0$:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\phi} = -\frac{qa^2}{m\omega} f'(\phi), \quad \Rightarrow \quad \alpha(\phi) = \alpha_0 - \frac{qa^2}{m\omega} f(\phi). \tag{4.256}$$

(d) Substituting into the equation:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\phi} = \frac{q}{m\omega} \left[k^\mu \alpha(\phi) - a^\mu \omega\right] f'(\phi). \tag{4.257}$$

(e) Integration yields:

$$u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} \left[k^\mu \int_{\phi_0}^{\phi} \alpha(\varphi) f'(\varphi) \mathrm{d}\varphi - a^\mu \omega \int_{\phi_0}^{\phi} f'(\varphi) \mathrm{d}\varphi \right] \tag{4.258}$$

$$= u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 \Delta f - \frac{qa^2}{2m\omega} \Delta(f^2) \right] - \frac{q}{m} a^\mu \Delta f, \tag{4.259}$$

where $\Delta f = f(\phi) - f(\phi_0)$, and $\Delta(f^2) = f(\phi)^2 - f(\phi_0)^2$.

(f) The trajectory is obtained by integrating once more:

$$x^\mu(\phi) = x^\mu(\phi_0) + \frac{1}{\omega} \int_{\phi_0}^{\phi} u^\mu(\varphi) \mathrm{d}\varphi. \tag{4.260}$$

Summary:

$$\boxed{\begin{aligned} & u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 \Delta f - \frac{qa^2}{2m\omega} \Delta(f^2) \right] - \frac{q}{m} a^\mu \Delta f, \\ & x^\mu(\phi) = x^\mu(\phi_0) + \frac{1}{\omega} \int_{\phi_0}^{\phi} u^\mu(\varphi)\, \mathrm{d}\varphi. \end{aligned}} \tag{4.261}$$

5. **Example: Sinusoidal Wave.**

Let

$$f(\phi) = \sin(\phi), \quad \Rightarrow \quad \int \sin(\phi) \mathrm{d}\phi = -\cos(\phi), \quad \int \sin^2(\phi) \mathrm{d}\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4}. \tag{4.262}$$

The velocity becomes:

$$u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 (\sin\phi - \sin\phi_0) - \frac{qa^2}{2m\omega} \left(\frac{\phi - \phi_0}{2} - \frac{\sin 2\phi - \sin 2\phi_0}{4} \right) \right]$$
$$- \frac{q}{m} a^\mu (\sin\phi - \sin\phi_0). \tag{4.263}$$

The integrated trajectory components are:

$$ct(\tau) = ct_0 + \frac{u^0(\tau_0)}{\omega} \Delta\phi + \frac{q}{m\omega^2} k^0 \left[\alpha_0 \Delta(-\cos\phi) - \frac{qa^2}{4m\omega} \left(\Delta\phi - \frac{\Delta(\sin 2\phi)}{2} \right) \right] - \frac{q}{m\omega} a^0 \Delta(-\cos\phi), \tag{4.264}$$

$$x^i(\tau) = x^i(\tau_0) + \frac{u^i(\tau_0)}{\omega} \Delta\phi + \frac{q}{m\omega^2} k^i \left[\alpha_0 \Delta(-\cos\phi) - \frac{qa^2}{4m\omega} \left(\Delta\phi - \frac{\Delta(\sin 2\phi)}{2} \right) \right] - \frac{q}{m\omega} a^i \Delta(-\cos\phi), \tag{4.265}$$

where

$$\phi = \omega\tau + \phi_0, \quad \alpha_0 = a_\mu u^\mu(\tau_0), \quad a^2 = a_\mu a^\mu. \tag{4.266}$$

This expression gives the full analytic trajectory of a charged particle in a monochromatic sinusoidal plane electromagnetic wave.

4.8.3 Field Theory

1. For an action depending on a field φ (scalar, tensor, etc.):

$$S = \int_\Omega \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)\, \mathrm{d}^4 x \tag{4.267}$$

We consider a variation $\varphi \mapsto \varphi + \varepsilon\eta$ with $\eta \in \mathcal{C}^1_c(\Omega)$ (compactly supported and continuously differentiable). Then:

$$\delta S = \frac{\mathrm{d}S}{\mathrm{d}\varepsilon}[\varphi + \varepsilon\eta]\bigg|_{\varepsilon=0} \tag{4.268}$$

$$= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi}\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \eta \right) \mathrm{d}^4 x \tag{4.269}$$

We integrate by parts the term $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \eta$:

$$u = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Rightarrow \mathrm{d}u = \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Big), \quad \mathrm{d}v = \partial_\mu \eta \Rightarrow v = \eta \tag{4.270}$$

We obtain:

$$\delta S = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right) \eta \, \mathrm{d}^4 x \tag{4.271}$$

Since η vanishes on the boundary $\partial\Omega$, we obtain the Euler-Lagrange equation for fields:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0} \tag{4.272}$$

2. We consider the electromagnetic action:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A^\mu j_\mu \tag{4.273}$$

We want to calculate:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}. \tag{4.274}$$

We start by using the chain rule:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\frac{1}{4} \left(\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A^\nu)} F^{\rho\sigma} + F_{\rho\sigma} \frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A^\nu)} \right). \tag{4.275}$$

Now, by differentiating the tensor $F_{\rho\sigma}$:

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A^\nu)} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho, \tag{4.276}$$

and similarly:

$$\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A^\nu)} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho, \tag{4.277}$$

thus:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\frac{1}{4} \left[\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) F^{\rho\sigma} + F_{\rho\sigma} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) \right] \tag{4.278}$$

$$= -\frac{1}{2} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) F^{\rho\sigma} \tag{4.279}$$

$$= -\frac{1}{2} (F^{\mu\nu} - F^{\nu\mu}) \tag{4.280}$$

$$= -F^{\mu\nu} \quad (\text{since } F^{\nu\mu} = -F^{\mu\nu}). \tag{4.281}$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = -F^{\mu\nu}} \tag{4.282}$$

Moreover,

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = -j^\mu \tag{4.283}$$

Therefore, by the Euler-Lagrange equation,

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu} \tag{4.284}$$

The homogeneous equations are obtained by noting that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ implies by construction[2]:

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0} \tag{4.285}$$

4.8.4 Trajectory of a Charged Particle in a Constant Magnetic Field

1. The field tensor $F^{\mu\nu}$: in the reference frame where $\mathbf{E} = 0$ and $\mathbf{B} = B\mathbf{e}_z$, we have:

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.286}$$

2. The equation of motion $\mathrm{d}u^\mu/\mathrm{d}\tau = (q/m)F^{\mu\nu}u_\nu$ implies $u^3 = \text{constant} = 0$, hence planar motion. The energy $E = \gamma mc^2$ is conserved since $F^{0\nu} = 0$.

3. The equation $\mathrm{d}u^\mu/\mathrm{d}\tau = (q/m)F^{\mu\nu}u_\nu$ gives:

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = (q/m)F^{12}u^2 = -(qB/m)u^2, \tag{4.287}$$

$$\frac{\mathrm{d}u^2}{\mathrm{d}\tau} = (q/m)F^{21}u^1 = (qB/m)u^1. \tag{4.288}$$

This describes uniform circular motion, thus:

$$\omega = \frac{qB}{m}, \qquad x(t) = R\cos\left(\frac{\omega}{\gamma}t\right), \quad y(t) = R\sin\left(\frac{\omega}{\gamma}t\right), \tag{4.289}$$

with γ constant, implying $\tau = t/\gamma$, and:

$$R = \frac{\gamma mv}{qB}. \tag{4.290}$$

4. Starting from:

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 a^2, \qquad E = \gamma mc^2, \quad a = \frac{v^2}{R}, \tag{4.291}$$

we obtain:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{q^2}{6\pi\varepsilon_0 c^5 m}\gamma^3 a^2 = -C(\gamma^2 - 1), \tag{4.292}$$

with:

$$C = \frac{q^4 B^2}{6\pi\varepsilon_0 c^5 m^3}. \tag{4.293}$$

[2] Simply differentiate each term, then substitute; everything cancels out.

5. Solving:
$$\frac{\mathrm{d}\gamma}{\gamma^2-1} = -C\mathrm{d}t \Rightarrow \frac{1}{2}\ln\left|\frac{\gamma-1}{\gamma+1}\right| = -Ct + C_0, \tag{4.294}$$
we find:
$$\boxed{\gamma(t) = \coth(Ct + C_0)}. \tag{4.295}$$
6. Using $\omega = \frac{qB}{m}$ and $v(t) = c\sqrt{1 - 1/\gamma(t)^2}$, we obtain:
$$R(t) = \frac{\gamma(t)mv(t)}{qB}, \quad \theta(t) = \int_0^t \frac{\omega}{\gamma(s)}\,\mathrm{d}s, \tag{4.296}$$
and then:
$$x(t) = R(t)\cos\theta(t), \tag{4.297}$$
$$y(t) = R(t)\sin\theta(t). \tag{4.298}$$
7. The trajectory spirals toward the origin since $R(t) \to 0$ and $\omega/\gamma(t) \to 0$, although oscillations persist. Numerically, this leads to error accumulation, requiring adaptive time steps to resolve the fast oscillations at early times.

4.8.5 Relativistic Collider Physics

We work in natural units, with $c = 1$.

1. The square of the total energy-momentum invariant is defined as:
$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2. \tag{4.299}$$
In the center-of-mass frame (CMS), the total available energy is:
$$E_{\text{tot}}^{(\text{CMS})} = \sqrt{s}. \tag{4.300}$$
2. For a head-on collision of two identical particles of mass m and energy E each (in the laboratory frame), the four-momenta are:
$$p_1 = (E, \mathbf{p}), \tag{4.301}$$
$$p_2 = (E, -\mathbf{p}), \tag{4.302}$$
which gives:
$$s = (p_1 + p_2)^2 = 2m^2 + 2(E^2 - \mathbf{p}^2) = 4E^2, \tag{4.303}$$
using $E^2 - \mathbf{p}^2 = m^2$ and $E \gg m$. Therefore:
$$\boxed{\sqrt{s} = 2E}. \tag{4.304}$$
3. For a fixed-target collision:
$$p_1 = (E_{\text{lab}}, \mathbf{p}), \tag{4.305}$$
$$p_2 = (m, 0), \tag{4.306}$$
then:
$$s = (p_1 + p_2)^2 = m^2 + m^2 + 2mE_{\text{lab}} = 2m^2 + 2mE_{\text{lab}}, \tag{4.307}$$

hence:

$$\boxed{s = 2m^2 + 2mE_{\text{lab}}}. \tag{4.308}$$

At the threshold for producing two new particles of mass m, we set $\sqrt{s} = 2m$, yielding:

$$2m = \sqrt{2m^2 + 2mE_{\text{lab}}} \Rightarrow E_{\text{lab}} = 2m. \tag{4.309}$$

4. To produce a single particle of mass M at threshold in a fixed-target experiment:

$$s = M^2 = m^2 + m^2 + 2mE_{\text{lab}} \Rightarrow E_{\text{lab}} = \frac{M^2 - 2m^2}{2m}. \tag{4.310}$$

In contrast, in a symmetric collider:

$$E_{\text{CM}} = \sqrt{s} = 2E = 2M \Rightarrow E = M. \tag{4.311}$$

Thus, for the same center-of-mass energy, the required lab-frame energy is much larger in a fixed-target setup than in a symmetric collider. This is why head-on colliders are more efficient for producing heavy particles at high energy.

4.9 Relativistic hydrodynamics

4.9.1 Classical Hydrodynamics

1. Consider a fluid region of fixed and closed volume V. The total mass is

$$M(t) = \int_V \rho(\boldsymbol{x}, t)\,\mathrm{d}V. \tag{4.312}$$

Mass conservation implies that the time variation of this mass is compensated by the mass flux leaving through the boundary ∂V:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\int_{\partial V} \rho\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{4.313}$$

where $\boldsymbol{n}$ is the outward normal vector of the surface.

By the divergence theorem (Gauss), this can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\boldsymbol{v})\,\mathrm{d}V. \tag{4.314}$$

Since V is fixed, one can interchange the time derivative and the integral:

$$\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\boldsymbol{v})\,\mathrm{d}V. \tag{4.315}$$

By arbitrariness of the volume V, we obtain the local equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{4.316}$$

If the fluid is incompressible, the mass density is constant and independent of t and $\boldsymbol{x}$: $\rho = \rho_0$. Then,

$$\frac{\partial\rho}{\partial t} = 0, \quad \nabla\rho = \boldsymbol{0}, \tag{4.317}$$

and (4.316) becomes

$$0 + \rho_0\nabla\cdot\boldsymbol{v} = 0 \implies \nabla\cdot\boldsymbol{v} = 0. \tag{4.318}$$

So for an incompressible fluid, the velocity field is divergence-free.

2. We start from Newton's second law applied to an elementary fluid particle of mass density ρ.

The left-hand side of the equation is the total acceleration multiplied by the mass density:

$$\rho\frac{D\boldsymbol{v}}{Dt} = \rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right), \tag{4.319}$$

where $\frac{D}{Dt}$ is the material derivative (following the particle).

On the right-hand side, we have volumetric forces:

- Force due to pressure gradient, which pushes the fluid from high pressure regions to low pressure: $-\nabla p$.

- Volumetric gravitational force: $\rho\boldsymbol{g}$.

The dynamic equilibrium writes as

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)=-\nabla p+\rho\boldsymbol{g}. \tag{4.320}$$

Physical meaning: - $\rho\frac{\partial \boldsymbol{v}}{\partial t}$: local variation of velocity, - $\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$: spatial variation linked to transport by fluid motion, - $-\nabla p$: internal force related to pressure gradients, - $\rho\boldsymbol{g}$: external gravitational force.

3. Viscous forces correspond to dissipative effects linked to fluid deformations. These forces are modeled by terms proportional to second derivatives of velocity components.

   The viscous stress tensor, in a Newtonian fluid, is proportional to the strain rate tensor. The volumetric contribution of the viscous force is

$$\boldsymbol{f}_{\text{visc}}=\eta\nabla^2\boldsymbol{v}+\left(\zeta+\frac{\eta}{3}\right)\nabla(\nabla\cdot\boldsymbol{v}), \tag{4.321}$$

   where η is the dynamic (shear) viscosity and ζ the bulk (volume) viscosity.

   Thus, the complete equation becomes

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)=-\nabla p+\eta\nabla^2\boldsymbol{v}+\left(\zeta+\frac{\eta}{3}\right)\nabla(\nabla\cdot\boldsymbol{v})+\rho\boldsymbol{g}. \tag{4.322}$$

   Role of viscous terms:

   - $\eta\nabla^2\boldsymbol{v}$ tends to smooth velocity gradients (momentum diffusion).

   - $\left(\zeta+\frac{\eta}{3}\right)\nabla(\nabla\cdot\boldsymbol{v})$ acts when the fluid is compressible, dissipating energy related to compression or dilation.

4. In the Lagrangian description, one follows the trajectory $\boldsymbol{X}(t)$ of an individual fluid particle. The total time derivative is then the derivative following this particle:

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}\cdot\nabla=\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla, \tag{4.323}$$

   since $\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}=\boldsymbol{v}(\boldsymbol{X},t)$.

   This formula defines the material or substantial derivative in the Eulerian description.

   The Eulerian description analyzes the fields $\boldsymbol{v}(\boldsymbol{x},t)$ at each fixed point in space, without following individual particle trajectories.

5. Consider a streamline parameterized by $s\mapsto\boldsymbol{r}(s)$ at fixed time t. The condition that the tangent is colinear to the velocity field writes

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\alpha(s)\boldsymbol{v}(\boldsymbol{r}(s),t), \tag{4.324}$$

   with $\alpha(s)$ a positive function.

   Choosing the parameter s such that $\alpha(s)=1$ (arclength parameterization or other), then

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\boldsymbol{v}(\boldsymbol{r}(s),t). \tag{4.325}$$

   Thus, streamlines are integral trajectories of the field $\boldsymbol{v}$ at instant t.

6. For a perfect, incompressible, stationary, and non-viscous fluid subject to a conservative force field with potential Φ, we start from the stationary Euler equation

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \rho \nabla \Phi. \tag{4.326}$$

We use the vector identity

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla \left(\frac{v^2}{2} \right) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}). \tag{4.327}$$

For an irrotational fluid (without vortices), $\nabla \times \boldsymbol{v} = 0$, so

$$\rho \nabla \left(\frac{v^2}{2} \right) = -\nabla p + \rho \nabla \Phi. \tag{4.328}$$

Rearranging:

$$\nabla \left(\frac{1}{2}\rho v^2 + p + \rho \Phi \right) = \mathbf{0}. \tag{4.329}$$

Thus, the quantity inside the parentheses is constant along a streamline:

$$\frac{1}{2}\rho v^2 + p + \rho \Phi = \text{constant}. \tag{4.330}$$

This is Bernoulli's theorem.

4.9.2 Introduction to Relativistic Hydrodynamics

1. The energy-momentum tensor of a perfect fluid is written as

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p\, g^{\mu\nu}, \tag{4.331}$$

where ε is the rest-frame energy density of the fluid, p the pressure, u^μ the four-velocity of the fluid (normalized by $u^\mu u_\mu = -1$), and $g^{\mu\nu}$ the Minkowski metric with signature $(- + ++)$.

(a) **Symmetry of $T^{\mu\nu}$**
The terms are built from symmetric products: - $u^\mu u^\nu$ is manifestly symmetric under $\mu \leftrightarrow \nu$. - $g^{\mu\nu}$ is symmetric.
Therefore, $T^{\mu\nu} = T^{\nu\mu}$.

(b) **Calculation in the rest frame**
In the fluid rest frame, the four-velocity is

$$u^\mu = (1, 0, 0, 0). \tag{4.332}$$

The energy-momentum tensor here, with signature $(- + ++)$,

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{4.333}$$

with

$$g^{\mu\nu} = \text{diag}(-1, +1, +1, +1). \tag{4.334}$$

Let us compute the component T^{00} explicitly:

$$T^{00} = (\varepsilon + p)u^0 u^0 + pg^{00} \tag{4.335}$$
$$= (\varepsilon + p) \times 1 \times 1 + p \times (-1) \tag{4.336}$$
$$= \varepsilon + p - p = \varepsilon. \tag{4.337}$$

Similarly, for spatial components $i, j = 1, 2, 3$:

$$T^{ij} = (\varepsilon + p)u^i u^j + pg^{ij} \tag{4.338}$$
$$= 0 + p\delta^{ij} = p\delta^{ij}. \tag{4.339}$$

Finally, for mixed components T^{0i}:

$$T^{0i} = (\varepsilon + p)u^0 u^i + pg^{0i} = 0, \tag{4.340}$$

since $u^i = 0$ and $g^{0i} = 0$.

(c) **Physical interpretation**

- $T^{00} = \varepsilon$ represents the total energy density in the fluid.
- $T^{0i} = 0$ means there is no energy flux in the rest frame (fluid at rest).
- $T^{ij} = p\delta^{ij}$ is the spatial stress tensor, here isotropic and equal to pressure p on the diagonal.

(d) **Calculation of the trace**

The trace of the tensor is obtained by contracting with the metric,

$$T^\mu_\mu = g_{\mu\nu} T^{\mu\nu} = g_{\mu\nu} \left[(\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} \right]. \tag{4.341}$$

This can be written as

$$T^\mu_\mu = (\varepsilon + p)g_{\mu\nu} u^\mu u^\nu + p\, g_{\mu\nu} g^{\mu\nu}. \tag{4.342}$$

Recall that

$$g_{\mu\nu} u^\mu u^\nu = u_\nu u^\nu = -1, \tag{4.343}$$

and that the contraction of the metric with itself is

$$g_{\mu\nu} g^{\mu\nu} = \mathrm{Tr}(\delta^\mu_\nu) = 4. \tag{4.344}$$

Substituting, we obtain

$$T^\mu_\mu = (\varepsilon + p)(-1) + p \times 4 \tag{4.345}$$
$$= -\varepsilon - p + 4p \tag{4.346}$$
$$= -\varepsilon + 3p. \tag{4.347}$$

(e) **Equation of state for an ultra-relativistic gas**

Recall that the energy density and pressure are expressed in statistical physics (within a volume V) by

$$\varepsilon = \frac{1}{V} \int \frac{\mathrm{d}^3 \|\mathbf{p}\|}{(2\pi)^3} E(\mathbf{p}) f(\mathbf{p}), \tag{4.348}$$

$$p = \frac{1}{3V} \int \frac{\mathrm{d}^3 \|\mathbf{p}\|}{(2\pi)^3} \frac{\|\mathbf{p}\|^2}{E(\mathbf{p})} f(\mathbf{p}), \tag{4.349}$$

where $f(\mathbf{p})$ is the distribution function.

In the ultra-relativistic regime, we consider particles that are almost massless, so

$$E(\mathbf{p}) = \sqrt{\|\mathbf{p}\|^2 + m^2} \simeq \|\mathbf{p}\|. \tag{4.350}$$

Substituting this approximation into equation (4.349):

$$p = \frac{1}{3V}\int \frac{\mathrm{d}^3\|\mathbf{p}\|}{(2\pi)^3}\frac{\|\mathbf{p}\|^2}{\|\mathbf{p}\|}f(\mathbf{p}) = \frac{1}{3V}\int \frac{\mathrm{d}^3\|\mathbf{p}\|}{(2\pi)^3}\|\mathbf{p}\|f(\mathbf{p}). \tag{4.351}$$

Now compare to the definition (4.348):

$$\varepsilon = \frac{1}{V}\int \frac{\mathrm{d}^3\|\mathbf{p}\|}{(2\pi)^3}\|\mathbf{p}\|f(\mathbf{p}). \tag{4.352}$$

We clearly see that

$$p = \frac{\varepsilon}{3}. \tag{4.353}$$

Consequence on the trace:

Replacing in the previously calculated trace,

$$T^{\mu}_{\mu} = -\varepsilon + 3p = -\varepsilon + 3 \times \frac{\varepsilon}{3} = 0. \tag{4.354}$$

This vanishing of the trace characterizes a conformal ultra-relativistic fluid.

2. The local conservation of the energy-momentum tensor reads

$$\partial_\mu T^{\mu\nu} = 0, \tag{4.355}$$

which represents four scalar equations expressing energy conservation (component $\nu = 0$) and momentum conservation ($\nu = 1, 2, 3$).

(a) The dynamic unknowns are the field variables describing the fluid state: the four-velocity u^μ (subject to the constraint $u_\mu u^\mu = -1$) and scalar thermodynamic variables such as the energy density ε, the pressure p (or temperature T).

(b) The system is not closed because there are more unknowns than equations. To solve it, an additional relation (equation of state) linking ε, p, and possibly T is needed. This relation stems from the microscopic physics or thermodynamics of the fluid.

3. At the local scale, classical thermodynamics can be applied to a comoving fluid element of infinitesimal volume $\mathrm{d}V$.

We introduce the corresponding extensive densities:

$$\varepsilon = \frac{U}{V}, \quad s = \frac{S}{V}, \quad n = \frac{N}{V}. \tag{4.356}$$

Within this framework, the first law of thermodynamics reads (in natural units, $c = 1$):

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}n. \tag{4.357}$$

We suppose, at first, that $\mu = 0$, i.e. that there is no particle number conservation (which is often a good approximation in a relativistic plasma where particles are created and annihilated).

Recall that the internal energy $U(S, V)$ is a homogeneous function of degree 1 in S and V (extensive variables). According to Euler's theorem for homogeneous functions:

$$U = TS - pV. \tag{4.358}$$

Dividing this relation by the volume V gives a relation between densities:

$$\frac{U}{V} = T\frac{S}{V} - p \quad \Rightarrow \quad \varepsilon = Ts - p. \tag{4.359}$$

One thus obtains the **Euler relation** for a relativistic fluid without chemical potential:

$$\varepsilon + p = Ts. \tag{4.360}$$

4. The speed of sound is defined by

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_s. \tag{4.361}$$

For an ultra-relativistic fluid where

$$p = \frac{\varepsilon}{3}, \tag{4.362}$$

we immediately get

$$c_s^2 = \frac{1}{3}. \tag{4.363}$$

Compared to the speed of light $c = 1$, we have

$$c_s = \frac{1}{\sqrt{3}} \simeq 0.577, \tag{4.364}$$

which is consistent since the speed of sound must be less than the speed of light.

4.9.3 Relativistic equation of motion

1. The conservation of the total particle number N in a relativistic fluid is globally written as the integral

$$N = \int_\Sigma j^\mu \, \mathrm{d}\Sigma_\mu, \tag{4.365}$$

where $j^\mu = nu^\mu$ is the particle current, with n the particle density in the comoving frame and u^μ the four-velocity of the fluid. Σ is a spacelike hypersurface oriented towards the future.

Local conservation is expressed by the condition that the integral is independent of the choice of Σ, which implies the local conservation law:

$$\partial_\mu j^\mu = 0. \tag{4.366}$$

This equation is explicitly written as

$$\partial_\mu (nu^\mu) = 0, \tag{4.367}$$

which is the covariant form of relativistic continuity.

Quick proof:

Let Σ_τ be a hypersurface of constant time τ. For

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = 0, \tag{4.368}$$

the local divergence of the current must vanish.

2. Starting from the conservation of the energy-momentum tensor,

$$\partial_\mu T^{\mu\nu} = 0, \tag{4.369}$$

with

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{4.370}$$

We want to obtain the relativistic equation of motion (relativistic Euler equation).

Developing:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left[(\varepsilon + p)u^\mu u^\nu\right] + \partial_\mu \left(pg^{\mu\nu}\right) = 0. \tag{4.371}$$

Since $g^{\mu\nu}$ is constant in flat spacetime,

$$\partial_\mu(pg^{\mu\nu}) = g^{\mu\nu}\partial_\mu p = \partial^\nu p, \tag{4.372}$$

because $g^{\mu\nu}$ raises indices.

Therefore,

$$\partial_\mu \left[(\varepsilon + p)u^\mu u^\nu\right] + \partial^\nu p = 0. \tag{4.373}$$

Applying the product rule to the first term:

$$u^\nu \partial_\mu[(\varepsilon + p)u^\mu] + (\varepsilon + p)u^\mu \partial_\mu u^\nu + \partial^\nu p = 0. \tag{4.374}$$

Projecting this equation onto u_ν gives an important relation.

Contract with u_ν:

$$u_\nu \partial_\mu \left[(\varepsilon + p)u^\mu u^\nu\right] + u_\nu \partial^\nu p = 0. \tag{4.375}$$

The first term becomes

$$u_\nu \partial_\mu[(\varepsilon + p)u^\mu u^\nu] = \partial_\mu[(\varepsilon + p)u^\mu(u_\nu u^\nu)] - (\varepsilon + p)u^\mu u_\nu \partial_\mu u^\nu. \tag{4.376}$$

Recall that

$$u_\nu u^\nu = -1, \tag{4.377}$$

which is constant.

Thus,

$$\partial_\mu[(\varepsilon + p)u^\mu(u_\nu u^\nu)] = \partial_\mu[-(\varepsilon + p)u^\mu] = -\partial_\mu[(\varepsilon + p)u^\mu]. \tag{4.378}$$

Therefore,

$$-\partial_\mu[(\varepsilon + p)u^\mu] - (\varepsilon + p)u^\mu u_\nu \partial_\mu u^\nu + u_\nu \partial^\nu p = 0. \tag{4.379}$$

But the term $u^\mu u_\nu \partial_\mu u^\nu$ can be shown to vanish because

$$u_\nu \partial_\mu u^\nu = \frac{1}{2}\partial_\mu(u_\nu u^\nu) = \frac{1}{2}\partial_\mu(-1) = 0. \tag{4.380}$$

Hence,

$$-\partial_\mu[(\varepsilon + p)u^\mu] + u_\nu \partial^\nu p = 0, \tag{4.381}$$

or equivalently,

$$\partial_\mu[(\varepsilon + p)u^\mu] = u^\nu \partial_\nu p. \tag{4.382}$$

Returning to the original equation and projecting orthogonally to u^ν using the projector

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{4.383}$$

we get the relativistic Euler equation:

$$(\varepsilon + p)u^\mu \partial_\mu u^\nu + \Delta^{\nu\mu}\partial_\mu p = 0. \tag{4.384}$$

Interpretation: The first term represents the relativistic inertia of the fluid, the second the pressure gradient projected perpendicular to the four-velocity.

4.9.4 Application to heavy-ion collisions

1. Bjorken coordinates are defined by

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2}\ln\frac{t+z}{t-z}. \tag{4.385}$$

Calculate the invariant interval element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ as a function of τ and η.

We have

$$t = \tau\cosh\eta, \quad z = \tau\sinh\eta. \tag{4.386}$$

Differentiating,

$$\mathrm{d}t = \cosh\eta\,\mathrm{d}\tau + \tau\sinh\eta\,\mathrm{d}\eta, \tag{4.387}$$

$$\mathrm{d}z = \sinh\eta\,\mathrm{d}\tau + \tau\cosh\eta\,\mathrm{d}\eta. \tag{4.388}$$

Calculate $-\mathrm{d}t^2 + \mathrm{d}z^2$:

$$-\mathrm{d}t^2 + \mathrm{d}z^2 = -(\cosh\eta\,\mathrm{d}\tau + \tau\sinh\eta\,\mathrm{d}\eta)^2 + (\sinh\eta\,\mathrm{d}\tau + \tau\cosh\eta\,\mathrm{d}\eta)^2. \tag{4.389}$$

Expanding,

$$= -\cosh^2\eta\,\mathrm{d}\tau^2 - 2\tau\cosh\eta\sinh\eta\,\mathrm{d}\tau\mathrm{d}\eta - \tau^2\sinh^2\eta\,\mathrm{d}\eta^2 \tag{4.390}$$

$$+ \sinh^2\eta\,\mathrm{d}\tau^2 + 2\tau\sinh\eta\cosh\eta\,\mathrm{d}\tau\mathrm{d}\eta + \tau^2\cosh^2\eta\,\mathrm{d}\eta^2. \tag{4.391}$$

The mixed terms $\mathrm{d}\tau\mathrm{d}\eta$ cancel, and using $\cosh^2\eta - \sinh^2\eta = 1$, we get

$$-\mathrm{d}t^2 + \mathrm{d}z^2 = -\mathrm{d}\tau^2 + \tau^2\mathrm{d}\eta^2. \tag{4.392}$$

Therefore, the interval becomes

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^2\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{4.393}$$

and the metric $g_{\mu\nu}$ in the (τ, x, y, η) basis is

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \tau^2). \tag{4.394}$$

2. In a boost-invariant fluid along z, one works in this curvilinear framework where the metric depends on τ. The conservation of the energy-momentum tensor becomes the covariant conservation

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0. \tag{4.395}$$

The Christoffel symbols are defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right). \tag{4.396}$$

In our metric, the only nonzero symbols are

$$\Gamma^\tau_{\eta\eta} = \tau, \quad \Gamma^\eta_{\tau\eta} = \Gamma^\eta_{\eta\tau} = \frac{1}{\tau}. \tag{4.397}$$

Consider the $\nu = \tau$ component of the conservation, since the energy evolution is related to this component.

Write

$$\nabla_\mu T^{\mu\tau} = 0. \tag{4.398}$$

Assuming a perfect fluid with transverse isotropy and boost invariance, and quantities depending only on τ, the equation reduces to

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{\varepsilon + p}{\tau} = 0. \tag{4.399}$$

Proof:

This expression comes from expanding $\nabla_\mu T^{\mu\tau}$, taking into account the dependence of $T^{\mu\nu}$ components and the nonzero Christoffel symbols. The term $\frac{\varepsilon+p}{\tau}$ corresponds to the geometric divergence linked to the longitudinal expansion in τ.

3. For $p = \varepsilon/3$, the equation

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{4}{3}\frac{\varepsilon}{\tau} = 0 \tag{4.400}$$

is an ordinary differential equation.

It is solved by separation of variables:

$$\frac{\mathrm{d}\varepsilon}{\varepsilon} = -\frac{4}{3}\frac{\mathrm{d}\tau}{\tau} \implies \ln\varepsilon = -\frac{4}{3}\ln\tau + \text{const}. \tag{4.401}$$

Hence,

$$\varepsilon(\tau) \propto \tau^{-\frac{4}{3}}. \tag{4.402}$$

Using the equation of state $\varepsilon \propto T^4$, one obtains

$$T(\tau) \propto \tau^{-\frac{1}{3}}. \tag{4.403}$$

4. The equation of state modeled during the quark-gluon plasma (QGP) $\to$ hadrons transition is

$$p = \frac{\varepsilon - 4B}{3}, \tag{4.404}$$

where B is the bag constant.

At the transition, the pressure vanishes, so $p = 0$, which implies

$$0 = \frac{\varepsilon - 4B}{3} \implies \varepsilon = 4B. \tag{4.405}$$

Assuming the relation $\varepsilon = aT^4$, the critical temperature T_c is obtained by

$$aT_c^4 = 4B \implies T_c = \left(\frac{4B}{a}\right)^{1/4}. \tag{4.406}$$

5. A nucleus modeled as a sphere of radius R has an approximate geometric cross section

$$\sigma \simeq \pi(2R)^2 = 4\pi R^2. \tag{4.407}$$

This cross section represents the effective transverse area for two nuclei to collide.

A central collision corresponds to an impact parameter $b \simeq 0$ (the nuclei fully overlap), whereas a peripheral collision corresponds to $b \sim 2R$ (only part of the nuclei overlap).

6. The initial volumetric energy density ε_0 depends on the degree of overlap of the nuclei; it is maximal in a central collision because the deposited energy density is higher.

Assuming

$$\varepsilon = aT^4, \tag{4.408}$$

where a is the Stefan-Boltzmann constant (specific to the quark-gluon plasma),

$$T_0 = \left(\frac{\varepsilon_0}{a}\right)^{1/4}. \tag{4.409}$$

For $\varepsilon_0 \sim 10\,\mathrm{GeV/fm}^3$, and taking a adapted to the QGP, one can estimate the initial temperature reached in RHIC collisions.

4.10 Hydrogen atom and radial equation

4.10.1 Separation of variables and radial equation

1. **Separation of variables**

   The Hamiltonian of the hydrogen atom, in the spherical basis, is written as:

$$\mathbf{H} = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{r}. \tag{4.410}$$

   In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}, \tag{4.411}$$

   where $\mathbf{L}^2$ is the square of the orbital angular momentum.

   We look for a solution of the form

$$\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi), \tag{4.412}$$

   where $Y_{\ell m}$ are the spherical harmonics simultaneous eigenfunctions of $\mathbf{L}^2$ and $\mathbf{L}_z$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}, \quad \mathbf{L}_z Y_{\ell m} = \hbar m Y_{\ell m}. \tag{4.413}$$

   Injecting into the stationary Schrödinger equation $\mathbf{H}\psi = E\psi$, we get the following radial equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER. \tag{4.414}$$

   Expanding the radial derivative,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r}, \tag{4.415}$$

   which gives the announced equation:

$$-\frac{\hbar^2}{2m_e}\left(\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right) - \frac{e^2}{r}R = ER. \tag{4.416}$$

2. **Change of function:** $u(r) = rR(r)$

   Putting $u(r) = rR(r)$, we calculate:

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2}, \tag{4.417}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} = \frac{1}{r}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - \frac{2}{r^2}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r^3}. \tag{4.418}$$

   Replacing in the radial equation, the terms in u/r^3 cancel and we get:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu. \tag{4.419}$$

3. **Dimensionless change of variable**

   We define
$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}, \quad \rho = \kappa r. \tag{4.420}$$

   The equation becomes
$$-\frac{\hbar^2}{2m_e}\kappa^2\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu. \tag{4.421}$$

   Since $E = -|E|$, dividing the whole equation by $-\frac{\hbar^2\kappa^2}{2m_e}$:
$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{2m_e e^2}{\hbar^2\kappa}\frac{1}{\rho} + 1\right]u. \tag{4.422}$$

   We then set
$$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa}. \tag{4.423}$$

   This gives the announced equation:
$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u. \tag{4.424}$$

4. **Ansatz on the form of $u(\rho)$**

   We set
$$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho). \tag{4.425}$$

   By calculating the second derivative of $u(\rho)$ and replacing into the differential equation, one finds that $v(\rho)$ satisfies:
$$\rho\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2} + (2\ell+2-\rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0. \tag{4.426}$$

   This equation is that of the confluent hypergeometric function.

5. **Power series and termination condition**

   We develop
$$v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k. \tag{4.427}$$

   The equation gives a recurrence relation between coefficients c_k. Generally, this series diverges as $\rho \to \infty$ unless the series is a polynomial, i.e. it stops at some finite order $\hat{k}$. The termination condition is
$$\rho_0 = 2n, \tag{4.428}$$
   where
$$n = \hat{k} + \ell + 1 \in \mathbb{N}^*. \tag{4.429}$$

6. **Expression of bound energy levels**

   Reinjecting the definition of ρ_0,
$$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa} = 2n \quad \Rightarrow \quad \kappa = \frac{m_e e^2}{2\hbar^2 n}. \tag{4.430}$$

Now

$$E = -\frac{\hbar^2\kappa^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2} \cdot \frac{1}{n^2}. \tag{4.431}$$

These are the quantized energy levels of the hydrogen atom.

7. **Degree of degeneracy**

For a level n, possible values of ℓ are

$$\ell = 0, 1, 2, \dots, n-1, \tag{4.432}$$

and for each ℓ, the values of m go from

$$m = -\ell, -\ell+1, \dots, \ell-1, \ell, \tag{4.433}$$

i.e. $(2\ell+1)$ values.

The degree of degeneracy is therefore

$$g_n = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1}1 = 2\frac{(n-1)n}{2} + n = n^2. \tag{4.434}$$

Interpretation: In this non-relativistic model without spin-orbit interactions or relativistic effects, the energy depends only on the principal quantum number n. This reflects the larger symmetry of the problem (rotational invariance and Runge-Lenz-type symmetry), which leads to this high degeneracy.

4.10.2 Ground state ($n = 1$) and radial properties

7. For $n = 1$, $\ell = 0$, $n_r = 0$:

$$u(r) = Are^{-r/a_0}, \quad \Rightarrow R(r) = \frac{u(r)}{r} = Ae^{-r/a_0}. \tag{4.435}$$

Normalization requires:

$$\int_0^\infty |R(r)|^2 r^2 \mathrm{d}r = |A|^2 \int_0^\infty e^{-2r/a_0} r^2 \mathrm{d}r = 1. \tag{4.436}$$

The integral yields $2!(a_0/2)^3 = a_0^3/4 \Rightarrow |A|^2 = 4/a_0^3$. Therefore:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}. \tag{4.437}$$

The spherical harmonic is $Y_{00} = 1/\sqrt{4\pi}$, so:

$$\psi_{1,0,0}(r,\theta,\phi) = \frac{2}{a_0^{3/2}} e^{-r/a_0} \cdot \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}. \tag{4.438}$$

The normalization is indeed satisfied:

$$\int |\psi|^2 \mathrm{d}^3x = \int_0^\infty |R|^2 r^2 \mathrm{d}r \int |Y|^2 \mathrm{d}\Omega = 1. \tag{4.439}$$

8. The radial probability density is:

$$P(r) = 4\pi|R(r)|^2r^2 = 4\pi\left(\frac{2}{a_0^{3/2}}\right)^2 e^{-2r/a_0}r^2 = \frac{16\pi}{a_0^3}r^2e^{-2r/a_0}. \tag{4.440}$$

It vanishes at $r = 0$ and as $r \to \infty$; the maximum is found at $r = a_0$. *Interpretation:* the most probable location to find the electron is at the Bohr radius.

9. We use:

$$\int_0^\infty r^n e^{-2r/a_0}\mathrm{d}r = n!\left(\frac{a_0}{2}\right)^{n+1}. \tag{4.441}$$

For $\langle r\rangle$:

$$\langle r\rangle = \int_0^\infty r|R(r)|^2r^2\mathrm{d}r = \frac{4}{a_0^3}\int_0^\infty r^3e^{-2r/a_0}\mathrm{d}r = \frac{4}{a_0^3}\cdot 3!\left(\frac{a_0}{2}\right)^4 = \frac{3}{2}a_0. \tag{4.442}$$

For $\langle r^2\rangle$:

$$\int r^4e^{-2r/a_0}\mathrm{d}r = 4!(a_0/2)^5 = 24(a_0/2)^5 \Rightarrow \langle r^2\rangle = 3a_0^2. \tag{4.443}$$

Thus:

$$(\Delta r)^2 = 3a_0^2 - (3a_0/2)^2 = 3a_0^2 - \frac{9}{4}a_0^2 = \frac{3}{4}a_0^2. \tag{4.444}$$

10. The Fourier transform of the ground state yields an isotropic distribution centered at $p = 0$. The expectation value is $\langle\mathbf{p}\rangle = 0$ (even function), and:

$$\langle p^2\rangle = \int\tilde{\psi}^*(\mathbf{p})p^2\tilde{\psi}(\mathbf{p})\mathrm{d}^3p. \tag{4.445}$$

It can be related to the average kinetic energy:

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = -E_1 = \frac{1}{2}E_0. \Rightarrow \langle p^2\rangle = m_eE_0. \tag{4.446}$$

11. The $1/n^2$ dependence explains the structure of the spectral lines described by the Rydberg formula:

$$\frac{1}{\lambda} = \mathcal{R}_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right). \tag{4.447}$$

The principal quantum number n orders the energy levels. In non-relativistic QM, ℓ does not affect E_n, unlike the relativistic case (Lamb shift, spin-orbit coupling).

4.11 Towards a relativistic formalism △

4.12 Pöschl–Teller potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$

1. The time-independent Schrödinger equation reads:

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{V_0}{\cosh^2(\alpha x)}\psi(x) = E\psi(x). \tag{4.448}$$

2. Let $u = \tanh(\alpha x)$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(1-u^2)$.

$$\begin{aligned}
\psi'(x) &= \frac{\mathrm{d}\phi}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u},\\
\psi''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u}\right)\\
&= \alpha\left(-2u\alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u} + (1-u^2)\alpha(1-u^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2}\right)\\
&= \alpha^2\left((1-u^2)^2\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u}\right).
\end{aligned}$$

3. The equation becomes:

$$-\frac{\hbar^2}{2m}\alpha^2\left((1-u^2)^2\phi'' - 2u(1-u^2)\phi'\right) - V_0(1-u^2)\phi = E\phi. \tag{4.449}$$

Divide by $(1-u^2)$ and set:

$$\lambda(\lambda+1) = \frac{2mV_0}{\hbar^2\alpha^2}, \qquad \mu^2 = -\frac{2mE}{\hbar^2\alpha^2}, \tag{4.450}$$

which gives:

$$(1-u^2)\phi'' - 2u\phi' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]\phi = 0. \tag{4.451}$$

4. We look for a solution of the form

$$\phi(u) = (1-u^2)^{\mu/2}P(u), \tag{4.452}$$

where $P(u)$ is a smooth function on $]-1,1[$, denoted simply as $Q(u) := (1-u^2)^{\mu/2}$ for brevity.

We then compute the derivatives of ϕ using the product rule:

$$\phi'(u) = Q'(u)P(u) + Q(u)P'(u), \tag{4.453}$$

$$\phi''(u) = Q''(u)P(u) + 2Q'(u)P'(u) + Q(u)P''(u). \tag{4.454}$$

Substitute into the differential equation from the previous question:

$$(1-u^2)\phi''(u) - 2u\,\phi'(u) + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]\phi(u) = 0. \tag{4.455}$$

Compute explicitly $Q'(u)$ and $Q''(u)$. We have:

$$Q(u) = (1-u^2)^{\mu/2}, \quad \Rightarrow \quad Q'(u) = -\mu u(1-u^2)^{\frac{\mu}{2}-1}, \tag{4.456}$$

$$Q''(u) = -\mu(1-u^2)^{\frac{\mu}{2}-1} + \mu(\mu-2)u^2(1-u^2)^{\frac{\mu}{2}-2}. \tag{4.457}$$

Substituting into the full equation:

$$(1-u^2)\left[Q''(u)P(u) + 2Q'(u)P'(u) + Q(u)P''(u)\right] - 2u\left[Q'(u)P(u) + Q(u)P'(u)\right] + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]Q(u)P(u) = 0.$$

All terms contain a common factor $Q(u)$, which can be factored out since $Q(u) \neq 0$ on $]-1,1[$. This yields a differential equation for $P(u)$ only:

$$(1-u^2)P''(u) - 2(\mu+1)uP'(u) + [\lambda(\lambda+1) - \mu(\mu+1)]P(u) = 0. \tag{4.458}$$

This is a differential equation of the type associated with Jacobi (or generalized Legendre) polynomials, which have polynomial solutions under certain quantization conditions (see next question).

5. (a) We have:

$$\int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x = \int_{-1}^{1} |\phi(u)|^2 \frac{\mathrm{d}u}{1-u^2} = \int_{-1}^{1} |P(u)|^2 (1-u^2)^{\mu-1} \mathrm{d}u < \infty. \tag{4.459}$$

   This is the normalization condition.

   (b) Set $P(u) = \sum_{p=0}^{\infty} a_p u^p$. Substituting into the ODE gives:

$$a_{p+2} = \frac{p(p+2\mu+1) - C}{(p+2)(p+1)} a_p \underset{\infty}{\sim} \frac{p^2}{p^2} a_p, \tag{4.460}$$

   with $C = \lambda(\lambda+1) - \mu(\mu+1)$. So $a_{p+2} \sim a_p$ as $p \to \infty$.

   (c) Thus,

$$\exists c \in \mathbb{R}^*, \quad P(u) \underset{u\to 1,\, p\to\infty}{\sim} \sum c u^p = \frac{c}{1-u}, \tag{4.461}$$

   which implies:

$$|\phi(u)|^2 \sim \frac{1}{(1-u)^{2-\mu}}. \tag{4.462}$$

   This is integrable only if $\mu > 1$. For bound states we have $\mu > 0$, hence divergence occurs if $\mu \leq 1$. Conclusion: for the integral to converge, the series must terminate $\Rightarrow P$ is a polynomial.

6. As shown earlier, $P(u)$ is a polynomial of degree $n \in \mathbb{N}$, and the condition for series termination is

$$a_{n+2} = 0. \tag{4.463}$$

Using the recurrence relation:

$$a_{p+2} = \frac{p(p+2\mu+1) - \lambda(\lambda+1) + \mu(\mu+1)}{(p+2)(p+1)} a_p, \tag{4.464}$$

and applying it to $p = n$, gives:

$$n(n+2\mu+1) = \lambda(\lambda+1) - \mu(\mu+1). \tag{4.465}$$

Developing both sides:

$$\begin{aligned}&\text{LHS: } n(n + 2\mu + 1) = n^2 + 2n\mu + n,\\ &\text{RHS: } \lambda(\lambda + 1) - \mu(\mu + 1) = \lambda^2 + \lambda - \mu^2 - \mu.\end{aligned}$$

Gathering terms:

$$\mu^2 + (2n + 1)\mu + (n^2 + n - \lambda^2 - \lambda) = 0. \tag{4.466}$$

We get a quadratic equation in μ:

$$\mu^2 + (2n + 1)\mu + A = 0, \quad \text{where } A = n(n + 1) - \lambda(\lambda + 1). \tag{4.467}$$

Its discriminant is:

$$\Delta = (2n + 1)^2 - 4A = (2n + 1)^2 - 4(n(n + 1) - \lambda(\lambda + 1)). \tag{4.468}$$

Expand:

$$\begin{aligned}\Delta &= 4n^2 + 4n + 1 - 4n(n + 1) + 4\lambda(\lambda + 1)\\ &= (4n^2 + 4n + 1 - 4n^2 - 4n) + 4\lambda(\lambda + 1)\\ &= 1 + 4\lambda(\lambda + 1).\end{aligned}$$

Thus,

$$\Delta = (2\lambda + 1)^2. \tag{4.469}$$

So the discriminant is a perfect square, and the equation admits two real roots:

$$\mu_\pm = \frac{-(2n + 1) \pm (2\lambda + 1)}{2}. \tag{4.470}$$

We compute both:

$$\mu_1 = \frac{-(2n + 1) + (2\lambda + 1)}{2} = \lambda - n, \quad \mu_2 = \frac{-(2n + 1) - (2\lambda + 1)}{2} = -(\lambda + n + 1). \tag{4.471}$$

The only physically admissible solution is the first, since for a bound state $\mu > 0$ (because $E < 0 \Rightarrow \mu^2 > 0$).

Therefore, the termination condition gives:

$$\mu = \lambda - n, \quad \text{with } n \in \mathbb{N},\ n < \lambda. \tag{4.472}$$

7. From Eq. (4.450), we deduce:

$$E_n = -\frac{\hbar^2\alpha^2}{2m}(\lambda - n)^2, \quad n = 0, 1, \ldots, \lfloor\lambda\rfloor. \tag{4.473}$$

8. The number of bound states is $N = \lfloor\lambda\rfloor + 1$, a finite number.

9. Physically, the potential $V(x) = -V_0/\cosh^2(\alpha x)$ decays exponentially at infinity ($V(x) \sim -4V_0 e^{-2\alpha|x|}$), too rapidly to allow for an infinite number of bound states. This is a finite-width potential well: the particle can only be confined to a finite number of energy levels.

4.13 Electrodynamic Instability of the Classical Atom

4.13.1 Calculation of the braking and radiation force $\mathbf{F}_{\text{rad}}$.

1.
$$\mathrm{d}E_{\text{at}} = \mathrm{d}W = \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \implies \Delta E_{\text{at}} = \int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \tag{4.474}$$

2. Be careful, this energy variation is the opposite of the energy radiated during the same interval:

$$\mathrm{d}E_{\text{at}} = -P\,\mathrm{d}t = -\frac{2e^2a^2}{3c^3}\mathrm{d}t = -\frac{2e^2\dot{\mathbf{v}}^2}{3c^3}\mathrm{d}t \tag{4.475}$$

We obtain:

$$\Delta E_{\text{at}} = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dot{\mathbf{v}}^2\mathrm{d}t \tag{4.476}$$

3. Moreover, by integrating by parts and assuming quasi-periodicity:

$$\Delta E_{\text{at}} = \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\mathbf{v}} \cdot \mathbf{v}\mathrm{d}t \tag{4.477}$$

By comparison with 4.474, a candidate force is the **Abraham-Lorentz radiation braking force**:

$$\mathbf{F}_{\text{rad}} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}} \tag{4.478}$$

4. Now consider the **Thomson model**, in which the electron is bound to the origin by a harmonic restoring force. The equation of motion becomes:

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} + \frac{2e^2}{3c^3}\dddot{\mathbf{r}} \tag{4.479}$$

We look for a solution of the form $r(t) = \text{Re}\left[r(0)e^{i\omega t}\right]$. The perturbative expansion:

$$\omega = \omega_0\left[1 + a(\omega_0\tau) + \mathcal{O}((\omega_0\tau)^2)\right] \tag{4.480}$$

gives $a = \frac{1}{2}$, hence finally:

$$\mathbf{r}(t) = \mathbf{r}(0)e^{-\omega_0^2\tau t}\cos(\omega_0 t) \tag{4.481}$$

The motion is therefore a damped oscillator. The characteristic damping time, or typical lifetime of the atom in this model, is:

$$T_{\text{nat}} = \frac{1}{\omega_0^2\tau} \sim 10^{-8}\ \text{s} \tag{4.482}$$

The classical atom is thus fundamentally unstable: the electron spirals toward the nucleus, very slowly on the atomic (pseudo-period) scale, but very rapidly on the macroscopic scale.

4.13.2 Conceptual problems generated by the braking force $\mathbf{F}_{\text{rad}}$.

1. The equation to solve is:

$$\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}} = \frac{1}{m}\mathbf{F}(t) \tag{4.483}$$

whose general solution is:

$$\dot{\mathbf{v}}(t) = v(t_0)e^{(t-t_0)/\tau} - \frac{1}{m\tau}\int_{t_0}^{t} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \tag{4.484}$$

2. An unacceptable phenomenon, sometimes called *preacceleration of a charged particle*, appears: if $F = 0$, the above expression clearly shows that the acceleration diverges exponentially at large times.

3. One can formally eliminate the divergent solutions by taking $t_0 = +\infty$. This is a boundary condition that effectively eliminates the so-called "initial condition".

4. Taking $t_0 = +\infty$, we obtain:

$$\begin{aligned}\dot{\mathbf{v}}(t) &= -\frac{1}{m\tau}\int_{t}^{+\infty} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \\ &= -\frac{1}{m}\int_{t}^{+\infty} K(t-t')\mathbf{F}(t')\,\mathrm{d}t'\end{aligned} \tag{4.485}$$

with $K(t-t') = \frac{1}{\tau}e^{(t-t')/\tau}$.

This is the *regularized form*, all the more so since the limit of zero charge correctly reproduces the Lorentz Force Electrodynamics (LFE).

Indeed, in the limit $e \to 0$, we have $\tau \to 0$ and the kernel $K(t-t')$ tends to a Dirac delta function $\delta(t-t')$, yielding:

$$\dot{\mathbf{v}}(t) = \frac{1}{m}\mathbf{F}(t) \tag{4.486}$$

5. It is already apparent that the acceleration at instant t depends on future values of the force. This equation therefore violates the **principle of causality**. A change of variable makes this clear. Let $s = \frac{t'-t}{\tau}$, we get:

$$\dot{\mathbf{v}}(t) = -\frac{1}{m}\int_{0}^{+\infty} e^{-s}\mathbf{F}(t+\tau s)\,\mathrm{d}s \tag{4.487}$$

6. With a step force:

$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathbf{F}_0 & \text{if } t \geq 0 \end{cases} \tag{4.488}$$

we obtain:

$$t < 0: \quad \dot{\mathbf{v}}(t) = -\frac{1}{m\tau}\int_{0}^{+\infty} e^{(t-t')/\tau}\cdot 0\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m}e^{t/\tau} \tag{4.489}$$

$$t > 0: \quad \dot{\mathbf{v}}(t) = -\frac{\mathbf{F}_0}{m\tau}\int_{t}^{+\infty} e^{(t-t')/\tau}\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m} \tag{4.490}$$

4.14 Geodesics in an Optical Medium △

4.15 Bose-Einstein Condensation

1. (a) **Expression for the average occupation number $\langle n_\varepsilon \rangle$ according to Bose-Einstein statistics**

For a bosonic system at thermal equilibrium in the grand-canonical ensemble, the average number of particles in an energy state ε is given by the Bose-Einstein distribution. The partition function for this state is:

$$Z_\varepsilon = \sum_{n_\varepsilon=0}^{\infty} e^{-\beta n_\varepsilon(\varepsilon-\mu)} = \frac{1}{1-e^{-\beta(\varepsilon-\mu)}}, \tag{4.491}$$

where $\beta = \frac{1}{k_B T}$ and μ is the chemical potential (which must satisfy $\mu < \varepsilon$ for convergence).

The average number of particles in this state is:

$$\langle n_\varepsilon \rangle = \frac{1}{Z_\varepsilon} \sum_{n_\varepsilon=0}^{\infty} n_\varepsilon e^{-\beta n_\varepsilon(\varepsilon-\mu)}. \tag{4.492}$$

By differentiating Z_ε with respect to $\beta(\varepsilon-\mu)$, we find:

$$\langle n_\varepsilon \rangle = -\frac{1}{Z_\varepsilon}\frac{\partial Z_\varepsilon}{\partial[\beta(\varepsilon-\mu)]} = -\frac{\partial \ln Z_\varepsilon}{\partial[\beta(\varepsilon-\mu)]} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}. \tag{4.493}$$

(b) **Expression of the density of states $g(\varepsilon)$ for a non-relativistic free gas in a cubic box**

Consider a free particle gas in a volume V with periodic boundary conditions. The non-relativistic kinetic energy is:

$$\varepsilon = \frac{\hbar^2 k^2}{2m}, \tag{4.494}$$

where $k = |\mathbf{k}|$ is the wave vector magnitude.

The number of states with wave vectors inside a sphere of radius k is given by the quantization in k-space:

$$\mathcal{N}(k) = \frac{V}{(2\pi)^3} \times \frac{4\pi k^3}{3}. \tag{4.495}$$

Differentiating with respect to k gives the density of states in k:

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k} = \frac{V}{2\pi^2} k^2. \tag{4.496}$$

Using the change of variables $\varepsilon = \frac{\hbar^2 k^2}{2m}$, we have:

$$k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}, \quad \mathrm{d}k = \frac{m}{\hbar^2 k}\mathrm{d}\varepsilon. \tag{4.497}$$

Thus the density of states per unit energy is:

$$g(\varepsilon) = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}\varepsilon} = \frac{V}{2\pi^2}k^2 \times \frac{m}{\hbar^2 k} = \frac{V}{2\pi^2}\frac{m}{\hbar^2}k = \frac{V}{2\pi^2}\frac{m}{\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}}. \tag{4.498}$$

Simplifying yields:

$$g(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}. \tag{4.499}$$

2. **Formula for the total average number of particles**

   The total average number of particles is the integral over all energy levels of the average occupation weighted by the density of states:

$$\langle N\rangle = \int_0^{+\infty} \langle n_\varepsilon\rangle g(\varepsilon)\mathrm{d}\varepsilon = \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}\mathrm{d}\varepsilon, \tag{4.500}$$

   which matches the given expression.

3. **Chemical potential as a function of temperature and density**

   In a closed system with N particles and density $\rho = N/V$, the canonical and grand-canonical ensembles are equivalent, so we set $\langle N\rangle = N$.

   Using the substitution $x = \beta\varepsilon$ and defining $\varphi(T) = e^{\mu(T)/(k_BT)}$, the integral becomes:

$$\rho = \frac{N}{V} = \left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^x/\varphi(T)-1}\mathrm{d}x = \left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^{x/\varphi(T)}-1}\mathrm{d}x, \tag{4.501}$$

   which corresponds to equation (3.171).

4. **Behavior of the chemical potential $\mu(T)$**

   Since $\varphi(T) = e^{\mu(T)/(k_BT)}$ and $\mu(T) < 0$, we have $0 < \varphi(T) < 1$. The function

$$f(\varphi) = \int_0^{+\infty} \frac{x^{1/2}}{e^x/\varphi - 1}\mathrm{d}x \tag{4.502}$$

   is increasing in φ on $(0, 1)$. When T decreases, the prefactor $\left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2}$ decreases as well. To keep the equality, $\varphi(T)$ must increase, so $\mu(T)$ increases.

5. **Constraint on chemical potential and critical temperature**

   The chemical potential μ must be strictly less than the lowest energy level (taken here as zero, the ground state), so $\mu < 0$. At the limit $\mu \to 0^-$, we define the critical temperature T_{BE} by:

$$\rho = \left(\frac{2mk_BT_{\mathrm{BE}}}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^x - 1}\mathrm{d}x. \tag{4.503}$$

   Using the approximation (3.172), we get:

$$T_{\mathrm{BE}} = \frac{2\pi\hbar^2}{mk_B}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3}, \tag{4.504}$$

   where $\zeta(3/2) \simeq 2.612$ is the Riemann zeta function.

6. **Breakdown of equation (3.171) for $T \leq T_{\mathrm{BE}}$**

   For $T \leq T_{\mathrm{BE}}$, setting $\mu = 0$ no longer satisfies equation (3.171) because the integral saturates and cannot increase further. The issue is that the population of the ground state, which can contain a macroscopic fraction of particles (the condensate), has not been taken into account.

7. **Population of the ground state N_0**

   Isolating the population of the ground state, N_0, we write:

$$N = N_0 + \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta\varepsilon}-1}\mathrm{d}\varepsilon = N_0 + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon}-1}\mathrm{d}\varepsilon, \tag{4.505}$$

with $\mu = 0$. Therefore, the condensate fraction is:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BE}}}\right)^{3/2}. \tag{4.506}$$

The condensate fraction exists only for $T < T_{\text{BE}}$ and grows as temperature decreases.

8. **Grand potential $\mathcal{J}$ and pressure for $T \le T_{\text{BE}}$**

   For $T \le T_{\text{BE}}$, the grand potential is:

$$\frac{\mathcal{J}}{k_B T} = -\ln(1 + N_0) + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \varepsilon^{1/2} \ln\left(1 - e^{-\beta\varepsilon}\right) \mathrm{d}\varepsilon. \tag{4.507}$$

   In the thermodynamic limit, N_0 is very large so $\ln(1 + N_0) \simeq \ln N_0$, which becomes negligible at the intensive scale. The pressure $P = -\mathcal{J}/V$ is then:

$$P = \frac{k_B T}{\lambda_{\text{th}}^3} \int_0^{+\infty} \frac{x^{3/2}}{e^x - 1} \mathrm{d}x, \tag{4.508}$$

   where $\lambda_{\text{th}} = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$ is the thermal de Broglie wavelength. Using (3.175), we see the pressure is independent of N_0, depends only on T, and decreases with temperature.

4.16 Decay Chain

4.16.1 Physical modeling of the decay chain

1. The first nucleus N_1 decays spontaneously with decay constant λ_1, so:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1. \tag{4.509}$$

Each nucleus N_k (for $k \in [\![2, n-1]\!]$) is created from the decay of N_{k-1} and disappears by its own decay. Thus:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = -\lambda_k N_k + \lambda_{k-1} N_{k-1}, \quad \text{for } k \in [\![2, n]\!]. \tag{4.510}$$

2. For $n = 2$, the system is:

$$\begin{cases} \frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1, \\ \frac{\mathrm{d}N_2}{\mathrm{d}t} = -\lambda_2 N_2 + \lambda_1 N_1. \end{cases} \tag{4.511}$$

Solving:

$$N_1(t) = N_0 e^{-\lambda_1 t}, \tag{4.512}$$

Then:

$$N_2(t) = \int_0^t \lambda_1 N_1(s) e^{-\lambda_2 (t-s)} \mathrm{d}s = \lambda_1 N_0 \int_0^t e^{-\lambda_1 s} e^{-\lambda_2 (t-s)} \mathrm{d}s. \tag{4.513}$$

Explicitly:

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \quad (\text{for } \lambda_1 \neq \lambda_2). \tag{4.514}$$

3. We compute:

$$N_1(t) + N_2(t) = N_0 e^{-\lambda_1 t} + \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) = N_0 \left(e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \right). \tag{4.515}$$

Simplifying:

$$N_1(t) + N_2(t) = N_0 \left(\frac{\lambda_2 - \lambda_1 + \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} \right) = N_0. \tag{4.516}$$

This expresses conservation of the total number of nuclei.

4. $N_2(t)$ reaches a maximum when $\frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\lambda_2 N_2 + \lambda_1 N_1 = 0 \quad \Rightarrow \quad N_2 = \frac{\lambda_1}{\lambda_2} N_1(t). \tag{4.517}$$

Inserting $N_1(t) = N_0 e^{-\lambda_1 t}$ into the expression of $N_2(t)$, and solving for t yields:

$$t_{\max} = \frac{1}{\lambda_2 - \lambda_1} \ln \left(\frac{\lambda_2}{\lambda_1} \right). \tag{4.518}$$

4.16.2 Mathematical analysis

1. The matrix A is lower triangular with distinct diagonal coefficients $-\lambda_k$. The matrix A is diagonalizable over $\mathbb{R}$ with eigenvalues $\{-\lambda_1, \dots, -\lambda_{n-1}, 0\}$ since $\lambda_n = 0$.

   Proof:

   Consider the matrix $A \in \mathbb{R}^{n\times n}$ defined by:

$$A = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & \lambda_2 & -\lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & -\lambda_n \end{pmatrix}. \tag{4.519}$$

   We want to determine the eigenvalues of A, i.e., the roots of the characteristic polynomial $\chi_A(X) = \det(A - XI_n)$.

   By construction, the matrix $A - XI_n$ is lower tridiagonal, and we denote $D_n = \det(A - XI_n)$. We prove by induction on n that:

$$D_n = \prod_{k=1}^{n}(-\lambda_k - X). \tag{4.520}$$

   Base case $(n = 1)$. We have simply:

$$D_1 = \det(-\lambda_1 - X) = -\lambda_1 - X, \tag{4.521}$$

   which matches the formula.

   Inductive step. Assume the formula holds for $n - 1$, i.e.:

$$D_{n-1} = \prod_{k=1}^{n-1}(-\lambda_k - X). \tag{4.522}$$

   We develop $D_n = \det(A_n - XI_n)$ by Laplace expansion along the last row, or equivalently use the recurrence relation for a lower tridiagonal matrix of the form:

$$D_n = (-\lambda_n - X)D_{n-1}, \tag{4.523}$$

   since only the diagonal term and the term just above it appear in the expansion, and the upper term is zero due to the triangular form.

   Thus, we obtain:

$$D_n = (-\lambda_n - X)\prod_{k=1}^{n-1}(-\lambda_k - X) = \prod_{k=1}^{n}(-\lambda_k - X), \tag{4.524}$$

   which concludes the induction.

   Conclusion. The eigenvalues of A are therefore given by:

$$\boxed{-\lambda_1, -\lambda_2, \dots, -\lambda_n = 0}. \tag{4.525}$$

2. Since A is a constant matrix, the Cauchy-Lipschitz theorem guarantees the existence and uniqueness of a global solution for all $t \geq 0$.

3. Let $E(t) = \|\mathbf{N}(t)\|^2 = \sum_{k=1}^{n} N_k(t)^2$. Since $\mathbf{N}(t)$ is differentiable (solution of a linear ODE), $E(t)$ is differentiable as well.

   (a) We have

   $$\langle x|Ax\rangle = x^\top Ax = (x^\top Ax)^\top = \frac{1}{2}x^\top(A + A^\top)x = x^\top Sx = \langle x|Sx\rangle \,, \tag{4.526}$$

   where $S = \frac{A+A^\top}{2}$ is the symmetric part of A. This identity holds for all $x \in \mathbb{R}^n$.

   (b) The matrix S is symmetric and tridiagonal, with:

   $$S_{kk} = -\lambda_k, \quad S_{k,k+1} = S_{k+1,k} = \frac{\lambda_k}{2}, \quad \text{for } k = 1, \dots, n-1. \tag{4.527}$$

   Then for any vector $x = (x_1, \dots, x_n)$, we compute:

   $$\begin{aligned} x^\top Sx &= \sum_{k=1}^{n} S_{kk}x_k^2 + 2\sum_{k=1}^{n-1} S_{k,k+1}x_k x_{k+1} \\ &= \sum_{k=1}^{n}(-\lambda_k)x_k^2 + \sum_{k=1}^{n-1} \lambda_k x_k x_{k+1} \\ &= -\frac{1}{2}\sum_{k=1}^{n-1} \lambda_k(x_k - x_{k+1})^2 - \frac{\lambda_1}{2}x_1^2. \end{aligned}$$

   Since all $\lambda_k \geq 0$, this scalar product is always less than or equal to zero.

   (c) Applying this to $\mathbf{N}(t)$, we obtain:

   $$E'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{N}(t)\|^2 = 2\left\langle \mathbf{N}(t)\middle|\dot{\mathbf{N}}(t)\right\rangle = 2\left\langle \mathbf{N}(t)|A\mathbf{N}(t)\right\rangle = 2\left\langle \mathbf{N}(t)|S\mathbf{N}(t)\right\rangle \leq 0. \tag{4.528}$$

   Therefore, $E(t)$ is a non-increasing function, and the system is stable.

4. The dynamical system is given by:

   $$\dot{\mathbf{N}}(t) = A\mathbf{N}(t), \tag{4.529}$$

   where $A \in \mathcal{M}_n(\mathbb{R})$ is a chain-decay matrix with $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$, and $\lambda_n = 0$.

   The matrix A is triangular or diagonalizable over $\mathbb{R}$, with one eigenvalue equal to 0 (corresponding to the stable isotope) and all other eigenvalues satisfying $\mathrm{Re}(\lambda) \leq -\alpha$.

   We consider an operator norm (equivalent to the usual norm), adapted to the spectrum of A, such that:

   $$\forall x \in \mathbb{R}^n, \quad \|e^{At}x\| \leq Ce^{\mu t}\|x\|, \quad \text{with } \mu = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \mathrm{Sp}(A), \lambda \neq 0\}. \tag{4.530}$$

   By hypothesis, $\mu \leq -\alpha < 0$.

   Note that $\mathbf{N}_\infty \in \ker A$, hence $A\mathbf{N}_\infty = 0$, and we have:

   $$\mathbf{N}(t) - \mathbf{N}_\infty = e^{At}(\mathbf{N}_0 - \mathbf{N}_\infty). \tag{4.531}$$

   Taking norms:

   $$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \leq Ce^{-\alpha t}\|\mathbf{N}_0 - \mathbf{N}_\infty\| \leq Ce^{-\alpha t}\|\mathbf{N}_0\|. \tag{4.532}$$

   This proves exponential convergence toward the equilibrium.

5. Summing the differential system:

$$\sum_{k=1}^{n} \frac{\mathrm{d}N_k}{\mathrm{d}t} = -\sum_{k=1}^{n-1} \lambda_k N_k + \sum_{k=2}^{n} \lambda_{k-1} N_{k-1} = 0, \tag{4.533}$$

which implies:

$$\sum_{k=1}^{n} N_k(t) = \sum_{k=1}^{n} N_k(0), \quad \forall t \geq 0. \tag{4.534}$$

Thus, the total number of nuclei is conserved.

4.17 From the Principle of Least Action to Einstein's Equations

4.17.1 From classical geometry to Lorentzian geometry

1. **Geometry of a surface in $\mathbb{R}^3$**

   (a) **Parametrization and first fundamental form**
   Let a surface $\Sigma \subset \mathbb{R}^3$ be parametrized by

   $$\mathbf{X}(u, v) = \big(X(u, v), Y(u, v), Z(u, v)\big),$$

   where (u, v) are local coordinates on the surface.
   The infinitesimal line element $\mathrm{d}s$ along the surface is given by

   $$\mathrm{d}s^2 = \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X} = (\partial_u \mathbf{X}\, \mathrm{d}u + \partial_v \mathbf{X}\, \mathrm{d}v) \cdot (\partial_u \mathbf{X}\, \mathrm{d}u + \partial_v \mathbf{X}\, \mathrm{d}v),$$

   where $\partial_u \mathbf{X} = \frac{\partial \mathbf{X}}{\partial u}$ and $\partial_v \mathbf{X} = \frac{\partial \mathbf{X}}{\partial v}$.
   Expanding the dot product:

   $$\mathrm{d}s^2 = (\partial_u \mathbf{X} \cdot \partial_u \mathbf{X})\, \mathrm{d}u^2 + 2(\partial_u \mathbf{X} \cdot \partial_v \mathbf{X})\, \mathrm{d}u\, \mathrm{d}v + (\partial_v \mathbf{X} \cdot \partial_v \mathbf{X})\, \mathrm{d}v^2. \tag{4.535}$$

   We define the coefficients of the **first fundamental form**:

   $$E = \partial_u \mathbf{X} \cdot \partial_u \mathbf{X}, \quad F = \partial_u \mathbf{X} \cdot \partial_v \mathbf{X}, \quad G = \partial_v \mathbf{X} \cdot \partial_v \mathbf{X}.$$

   Thus, the line element can be written compactly as:

   $$\mathrm{d}s^2 = E\, \mathrm{d}u^2 + 2F\, \mathrm{d}u\, \mathrm{d}v + G\, \mathrm{d}v^2.$$

   The associated matrix is

   $$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

   which defines a scalar product on the tangent space of the surface at each point. Remark: g is symmetric, and for a regular surface we have $E, G > 0$ and $EG - F^2 > 0$, which ensures that the metric is positive definite.

   (b) **Example: the plane $\mathbb{R}^2$**
   Take $\mathbf{X}(x, y) = (x, y, 0)$. Then

   $$\partial_x \mathbf{X} = (1, 0, 0), \quad \partial_y \mathbf{X} = (0, 1, 0),$$

   so

   $$E = 1, \quad F = 0, \quad G = 1, \quad \Rightarrow \quad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2.$$

   (c) **Second fundamental form and curvature**
   The **second fundamental form** II measures the curvature of the surface in the ambient space. For a surface $\mathbf{X}(u, v)$ with unit normal vector $\mathbf{n}$, it is written as

   $$\mathrm{II} = L\, \mathrm{d}u^2 + 2M\, \mathrm{d}u\, \mathrm{d}v + N\, \mathrm{d}v^2,$$

   where

   $$L = \mathbf{X}_{uu} \cdot \mathbf{n}, \quad M = \mathbf{X}_{uv} \cdot \mathbf{n}, \quad N = \mathbf{X}_{vv} \cdot \mathbf{n}.$$

The Gaussian curvature is the product of the principal curvatures k_1, k_2, which are the eigenvalues of $g^{-1}\mathrm{II}$:

$$\kappa = k_1 k_2 = \frac{\det \mathrm{II}}{\det g}.$$

Thus, κ is an **intrinsic** quantity of the surface: it does not depend on how the surface is embedded in $\mathbb{R}^3$.

2. **Intrinsic definition of a metric on a manifold**

   (a) On a manifold $\mathcal{M}$, one can locally define the distance between points via a metric tensor $g_{\mu\nu}(x)$:

   $$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu,$$

   where $\mu, \nu = 1, \dots, n$ and $n = \dim \mathcal{M}$.

   (b) Example: for a surface in $\mathbb{R}^3$, $g_{\mu\nu}$ corresponds exactly to the matrix $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ defined above.

3. **Hyperbolic metric on the upper half-plane**

   (a) Define the hyperbolic metric:

   $$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \quad y > 0.$$

   The metric matrix and its determinant:

   $$g = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det g = \frac{1}{y^4}.$$

   (b) The Christoffel symbols:

   $$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^y_{yy} = -\frac{1}{y}, \quad \Gamma^x_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0.$$

   (c) Gaussian curvature: $\kappa = -1$ (constant and negative).

   (d) **Geodesics**

   Let a curve be parametrized by x, $y(x)$. The action (length) is:

   $$\mathcal{L} = \frac{\sqrt{1 + (y')^2}}{y}, \quad y' = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Since $\mathcal{L}$ does not depend explicitly on x, one can use Beltrami's identity:

   $$\mathcal{L} - y'\frac{\partial \mathcal{L}}{\partial y'} = \lambda \quad \Rightarrow \quad y\sqrt{1 + (y')^2} = \frac{1}{\lambda}.$$

   Solving for y':

   $$y' = \pm\sqrt{\frac{1}{\lambda^2 y^2} - 1}.$$

   Integrating, one obtains the equation of a circle:

   $$(x - x_0)^2 + y^2 = \frac{1}{\lambda^2}.$$

   Special case $\lambda = 0$: vertical geodesic $x = x_0$.

4.17.2 Pseudo-Riemannian manifolds and geodesics

1. **Lorentzian metrics**

   (a) A **pseudo-Riemannian manifold** is a manifold equipped with a symmetric metric tensor $g_{\mu\nu}(x)$ which is not necessarily positive definite. A **Lorentzian metric** is a pseudo-Riemannian metric of signature $(-,+,+,+)$ (1 time direction and 3 spatial directions), adapted to special and general relativity.
   Thus, for any curve $\gamma(\tau)$ on the manifold, one defines the "length" or interval:

   $$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu,$$

   which can be negative, zero, or positive, depending on the type of tangent vector:
   - $\mathrm{d}s^2 < 0$: **timelike** vector,
   - $\mathrm{d}s^2 = 0$: **null/lightlike** vector (trajectory of light),
   - $\mathrm{d}s^2 > 0$: **spacelike** vector.

   (b) Example: Minkowski metric

   $$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \quad \Rightarrow \quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

   Timelike vectors have negative norm, spacelike vectors have positive norm, and null vectors (such as the trajectory of a photon) have zero norm.

2. **Curves and geodesics**
   Let a curve $\gamma(t) = (x^0(t), x^1(t), x^2(t), x^3(t))$ on the manifold, parametrized by a parameter t (often the proper time τ for a massive particle).

   (a) The **length** (or action) of the curve is

   $$L[\gamma] = \int_{t_1}^{t_2} \sqrt{|g_{\mu\nu}(x)\,\dot{x}^\mu\dot{x}^\nu|}\,\mathrm{d}t, \tag{4.536}$$

   where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$.

   (b) The curves that **extremize** this action (minimize or maximize depending on the type) satisfy the **Euler-Lagrange equations**:

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda}\right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} = 0, \tag{4.537}$$

   with $\mathcal{L} = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$ (quadratic form often used to simplify computations, it gives the same geodesics).

   (c) Detailed development:
   For $\mathcal{L} = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$, we compute:

   $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda} = g_{\lambda\nu}\dot{x}^\nu, \tag{4.538}$$

   and

   $$\frac{\partial \mathcal{L}}{\partial x^\lambda} = \frac{1}{2}\partial_\lambda g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu. \tag{4.539}$$

   The Euler-Lagrange equations then give

   $$\frac{\mathrm{d}}{\mathrm{d}t}(g_{\lambda\nu}\dot{x}^\nu) - \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{4.540}$$

Expanding the derivative:

$$g_{\lambda\nu}\ddot{x}^\nu + (\partial_\sigma g_{\lambda\nu})\dot{x}^\sigma\dot{x}^\nu - \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

Multiplying by $g^{\lambda\kappa}$ to isolate $\ddot{x}^\kappa$:

$$\ddot{x}^\kappa + \Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0,$$

where the **Christoffel symbols** are

$$\Gamma^\kappa_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{4.541}$$

Thus, geodesics are the curves whose covariant acceleration vanishes.

(d) **Geodesic equations in a local inertial frame**

Consider a point p of spacetime and a local frame (ξ^α) centered at p (local inertial coordinates or "normal coordinates"). By definition, in this frame:

$$g_{\alpha\beta}(\xi = 0) = \eta_{\alpha\beta}, \quad \partial_\gamma g_{\alpha\beta}(\xi = 0) = 0, \tag{4.542}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.
The Christoffel symbols are defined as:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right). \tag{4.543}$$

In this local frame:

$$\partial_\beta g_{\gamma\delta} = 0 \quad \Rightarrow \quad \Gamma^\alpha_{\beta\gamma} = 0. \tag{4.544}$$

The general geodesic equation is:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{4.545}$$

In local coordinates (ξ^α), the Christoffels vanish:

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} + 0 \cdot \frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\rho}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0. \tag{4.546}$$

This equation describes uniform rectilinear motion, exactly as in special relativity.

Coordinate change back to the general metric x^μ:

If one performs a coordinate change $\xi^\alpha \mapsto x^\mu(\xi)$, then the second derivative transforms as:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = \frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} + \frac{\partial^2x^\mu}{\partial\xi^\alpha\partial\xi^\beta}\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\beta}{\mathrm{d}\tau}. \tag{4.547}$$

Since $\mathrm{d}^2\xi^\alpha/\mathrm{d}\tau^2 = 0$, it remains:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \underbrace{-\frac{\partial^2x^\mu}{\partial\xi^\alpha\partial\xi^\beta}\frac{\partial\xi^\alpha}{\partial x^\nu}\frac{\partial\xi^\beta}{\partial x^\rho}}_{\Gamma^\mu_{\nu\rho}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{4.548}$$

One recognizes the general definition of the Christoffel symbols under coordinate change.

Conclusion: Thus, geodesics in the general metric $g_{\mu\nu}(x)$ are given by

4.17.3 From classical geometry to Lorentzian geometry

1. **Geometry of a surface in $\mathbb{R}^3$**

   (a) **Parametrization and first fundamental form**
   Let a surface $\Sigma \subset \mathbb{R}^3$ be parametrized by

$$\mathbf{X}(u,v) = \big(X(u,v), Y(u,v), Z(u,v)\big),$$

   where (u,v) are local coordinates on the surface.
   The infinitesimal line element $\mathrm{d}s$ along the surface is given by

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} = (\partial_u\mathbf{X}\,\mathrm{d}u + \partial_v\mathbf{X}\,\mathrm{d}v)\cdot(\partial_u\mathbf{X}\,\mathrm{d}u + \partial_v\mathbf{X}\,\mathrm{d}v),$$

   where $\partial_u\mathbf{X} = \frac{\partial \mathbf{X}}{\partial u}$ and $\partial_v\mathbf{X} = \frac{\partial \mathbf{X}}{\partial v}$.
   Expanding the dot product:

$$\mathrm{d}s^2 = (\partial_u\mathbf{X}\cdot\partial_u\mathbf{X})\,\mathrm{d}u^2 + 2(\partial_u\mathbf{X}\cdot\partial_v\mathbf{X})\,\mathrm{d}u\,\mathrm{d}v + (\partial_v\mathbf{X}\cdot\partial_v\mathbf{X})\,\mathrm{d}v^2. \quad (4.549)$$

   We define the coefficients of the **first fundamental form**:

$$E = \partial_u\mathbf{X}\cdot\partial_u\mathbf{X}, \quad F = \partial_u\mathbf{X}\cdot\partial_v\mathbf{X}, \quad G = \partial_v\mathbf{X}\cdot\partial_v\mathbf{X}.$$

   Thus, the line element can be written compactly as:

$$\mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2.$$

   The associated matrix is

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

   which defines a scalar product on the tangent space of the surface at each point. Remark: g is symmetric, and for a regular surface we have $E, G > 0$ and $EG - F^2 > 0$, which ensures that the metric is positive definite.

   (b) **Example: the plane $\mathbb{R}^2$**
   Take $\mathbf{X}(x,y) = (x,y,0)$. Then

$$\partial_x\mathbf{X} = (1,0,0), \quad \partial_y\mathbf{X} = (0,1,0),$$

   so

$$E = 1, \quad F = 0, \quad G = 1, \quad \Rightarrow \quad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2.$$

   (c) **Second fundamental form and curvature**
   The **second fundamental form** II measures the curvature of the surface in the ambient space. For a surface $\mathbf{X}(u,v)$ with unit normal vector $\mathbf{n}$, it is written as

$$\mathrm{II} = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2,$$

   where

$$L = \mathbf{X}_{uu}\cdot\mathbf{n}, \quad M = \mathbf{X}_{uv}\cdot\mathbf{n}, \quad N = \mathbf{X}_{vv}\cdot\mathbf{n}.$$

   The Gaussian curvature is the product of the principal curvatures k_1, k_2, which are the eigenvalues of $g^{-1}\mathrm{II}$:

$$\kappa = k_1 k_2 = \frac{\det \mathrm{II}}{\det g}.$$

   Thus, κ is an **intrinsic** quantity of the surface: it does not depend on how the surface is embedded in $\mathbb{R}^3$.

2. **Intrinsic definition of a metric on a manifold**

   (a) On a manifold $\mathcal{M}$, one can locally define the distance between points via a metric tensor $g_{\mu\nu}(x)$:
   $$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu,$$
   where $\mu, \nu = 1, \ldots, n$ and $n = \dim \mathcal{M}$.

   (b) Example: for a surface in $\mathbb{R}^3$, $g_{\mu\nu}$ corresponds exactly to the matrix $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ defined above.

3. **Hyperbolic metric on the upper half-plane**

   (a) Define the hyperbolic metric:
   $$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \quad y > 0.$$
   The metric matrix and its determinant:
   $$g = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det g = \frac{1}{y^4}.$$

   (b) The Christoffel symbols:
   $$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^y_{yy} = -\frac{1}{y}, \quad \Gamma^x_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0.$$

   (c) Gaussian curvature: $\kappa = -1$ (constant and negative).

   (d) **Geodesics**
   Let a curve be parametrized by $x, y(x)$. The action (length) is:
   $$\mathcal{L} = \frac{\sqrt{1 + (y')^2}}{y}, \quad y' = \frac{\mathrm{d}y}{\mathrm{d}x}.$$
   Since $\mathcal{L}$ does not depend explicitly on x, one can use Beltrami's identity:
   $$\mathcal{L} - y'\frac{\partial \mathcal{L}}{\partial y'} = \lambda \quad \Rightarrow \quad y\sqrt{1 + (y')^2} = \frac{1}{\lambda}.$$
   Solving for y':
   $$y' = \pm\sqrt{\frac{1}{\lambda^2 y^2} - 1}.$$
   Integrating, one obtains the equation of a circle:
   $$(x - x_0)^2 + y^2 = \frac{1}{\lambda^2}.$$
   Special case $\lambda = 0$: vertical geodesic $x = x_0$.

4.17.4 Pseudo-Riemannian manifolds and geodesics

1. **Lorentzian metrics**

   (a) A **pseudo-Riemannian manifold** is a manifold equipped with a symmetric metric tensor $g_{\mu\nu}(x)$ which is not necessarily positive definite. A **Lorentzian metric** is a pseudo-Riemannian metric of signature $(-,+,+,+)$ (1 time direction and 3 spatial directions), adapted to special and general relativity.
   Thus, for any curve $\gamma(\tau)$ on the manifold, one defines the "length" or interval:
   $$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu,$$
   which can be negative, zero, or positive, depending on the type of tangent vector:
   - $\mathrm{d}s^2 < 0$: **timelike** vector,
   - $\mathrm{d}s^2 = 0$: **null/lightlike** vector (trajectory of light),
   - $\mathrm{d}s^2 > 0$: **spacelike** vector.

   (b) Example: Minkowski metric
   $$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \quad \Rightarrow \quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$
   Timelike vectors have negative norm, spacelike vectors have positive norm, and null vectors (such as the trajectory of a photon) have zero norm.

2. **Curves and geodesics**

   Let a curve $\gamma(t) = (x^0(t), x^1(t), x^2(t), x^3(t))$ on the manifold, parametrized by a parameter t (often the proper time τ for a massive particle).

   (a) The **length** (or action) of the curve is
   $$L[\gamma] = \int_{t_1}^{t_2} \sqrt{|g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu|}\,\mathrm{d}t, \tag{4.550}$$
   where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$.

   (b) The curves that **extremize** this action (minimize or maximize depending on the type) satisfy the **Euler-Lagrange equations**:
   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda}\right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} = 0, \tag{4.551}$$
   with $\mathcal{L} = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$ (quadratic form often used to simplify computations, it gives the same geodesics).

   (c) Detailed development:
   For $\mathcal{L} = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$, we compute:
   $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\lambda} = g_{\lambda\nu}\dot{x}^\nu, \tag{4.552}$$
   and
   $$\frac{\partial \mathcal{L}}{\partial x^\lambda} = \frac{1}{2}\partial_\lambda g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu. \tag{4.553}$$
   The Euler-Lagrange equations then give
   $$\frac{\mathrm{d}}{\mathrm{d}t}(g_{\lambda\nu}\dot{x}^\nu) - \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{4.554}$$

Expanding the derivative:

$$g_{\lambda\nu}\ddot{x}^{\nu} + (\partial_{\sigma}g_{\lambda\nu})\dot{x}^{\sigma}\dot{x}^{\nu} - \frac{1}{2}\partial_{\lambda}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0.$$

Multiplying by $g^{\lambda\kappa}$ to isolate $\ddot{x}^{\kappa}$:

$$\ddot{x}^{\kappa} + \Gamma^{\kappa}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0,$$

where the **Christoffel symbols** are

$$\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}\right). \tag{4.555}$$

Thus, geodesics are the curves whose covariant acceleration vanishes.

(d) **Geodesic equations in a local inertial frame**

Consider a point p of spacetime and a local frame (ξ^{α}) centered at p (local inertial coordinates or "normal coordinates"). By definition, in this frame:

$$g_{\alpha\beta}(\xi = 0) = \eta_{\alpha\beta}, \quad \partial_{\gamma}g_{\alpha\beta}(\xi = 0) = 0, \tag{4.556}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.
The Christoffel symbols are defined as:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_{\beta}g_{\gamma\delta} + \partial_{\gamma}g_{\beta\delta} - \partial_{\delta}g_{\beta\gamma}\right). \tag{4.557}$$

In this local frame:

$$\partial_{\beta}g_{\gamma\delta} = 0 \quad \Rightarrow \quad \Gamma^{\alpha}_{\beta\gamma} = 0. \tag{4.558}$$

The general geodesic equation is:

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} = 0. \tag{4.559}$$

In local coordinates (ξ^{α}), the Christoffels vanish:

$$\frac{\mathrm{d}^2\xi^{\alpha}}{\mathrm{d}\tau^2} + 0 \cdot \frac{\mathrm{d}\xi^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^{\rho}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\xi^{\alpha}}{\mathrm{d}\tau^2} = 0. \tag{4.560}$$

This equation describes uniform rectilinear motion, exactly as in special relativity.

Coordinate change back to the general metric x^{μ}:

If one performs a coordinate change $\xi^{\alpha} \mapsto x^{\mu}(\xi)$, then the second derivative transforms as:

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} = \frac{\partial x^{\mu}}{\partial\xi^{\alpha}}\frac{\mathrm{d}^2\xi^{\alpha}}{\mathrm{d}\tau^2} + \frac{\partial^2x^{\mu}}{\partial\xi^{\alpha}\partial\xi^{\beta}}\frac{\mathrm{d}\xi^{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^{\beta}}{\mathrm{d}\tau}. \tag{4.561}$$

Since $\mathrm{d}^2\xi^{\alpha}/\mathrm{d}\tau^2 = 0$, it remains:

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \underbrace{-\frac{\partial^2x^{\mu}}{\partial\xi^{\alpha}\partial\xi^{\beta}}\frac{\partial\xi^{\alpha}}{\partial x^{\nu}}\frac{\partial\xi^{\beta}}{\partial x^{\rho}}}_{\Gamma^{\mu}_{\nu\rho}}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} = 0. \tag{4.562}$$

One recognizes the general definition of the Christoffel symbols under coordinate change.

Conclusion: Thus, geodesics in the general metric $g_{\mu\nu}(x)$ are given by

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\nu\rho}(x)\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} = 0, \tag{4.563}$$

which corresponds to the trajectory of a free particle in a curved spacetime.

4.17.5 Curvature and the Einstein–Hilbert action

1. **Definition of curvature objects**

   Let a differentiable manifold be equipped with a metric $g_{\mu\nu}$. First one defines the **Levi-Civita connection**, which is the unique torsion-free connection compatible with the metric:

   $$\nabla_\lambda g_{\mu\nu} = 0, \quad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}.$$

   From the connection one defines the **Riemann tensor** $R^\rho{}_{\sigma\mu\nu}$ which measures the non-commutativity of covariant derivatives:

   $$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{4.564}$$

   The **Ricci tensor** is obtained by contracting the Riemann tensor:

   $$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \tag{4.565}$$

   The **scalar curvature** R is the trace of the Ricci tensor:

   $$R = g^{\mu\nu} R_{\mu\nu}. \tag{4.566}$$

2. **Case of a locally flat metric**

   If the metric is locally flat (i.e. there exists a frame where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu} = 0$), then all Christoffel symbols $\Gamma^\lambda_{\mu\nu} = 0$. Consequently, all first and second derivatives vanish, and therefore:

   $$R^\rho{}_{\sigma\mu\nu} = 0, \quad R_{\mu\nu} = 0, \quad R = 0.$$

3. **Covariant scalar action: the Einstein–Hilbert action**

   We seek an action $S[g]$ constructed from $g_{\mu\nu}$ and its derivatives up to second order, which is a scalar under coordinate transformations. The unique candidate (up to an overall multiplicative constant) is:

   $$S[g] = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x, \tag{4.567}$$

   where $g = \det(g_{\mu\nu})$ and G is the gravitational constant.

   This action is called the **Einstein–Hilbert action**.

4. **Variation of the action and Einstein equations**

   To obtain the equations of motion, vary $S[g]$ with respect to the metric $g^{\mu\nu}$:

   $$\delta S[g] = \frac{1}{16\pi G}\int \left(\delta R\sqrt{-g} + R\delta\sqrt{-g}\right)\mathrm{d}^4x. \tag{4.568}$$

   We use the following formulas:

   $$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{4.569}$$

   $$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}. \tag{4.570}$$

   Integrating by parts the covariant derivatives and discarding boundary terms yields the **vacuum Einstein equations**:

   $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{4.571}$$

5. **Adding matter**

   If one adds a matter term $S_{\text{matter}}[g, \psi]$, the total action is:

$$S_{\text{total}} = \frac{1}{16\pi G} \int R\sqrt{-g}\, \mathrm{d}^4x + S_{\text{matter}}[g, \psi], \tag{4.572}$$

   where ψ denotes the matter fields.

   The **energy–momentum tensor** is defined by:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{4.573}$$

   The variation of the total action then leads to the full **Einstein equations**:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{4.574}$$

6. **Physical interpretation**

   - The left-hand side, $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, describes the curvature of spacetime.
   - The right-hand side, $8\pi G T_{\mu\nu}$, describes the distribution of energy and matter.
   - Thus, gravity is interpreted as the curvature of spacetime produced by matter and energy.

4.17.6 Principle of least action and the Einstein equations

1. **Functional variation for a field**

   Let an action for a scalar or tensor field $\varphi(x)$ be:

$$S[\varphi] = \int \mathcal{L}(\varphi, \partial_\mu \varphi)\, \mathrm{d}^4x. \tag{4.575}$$

   Consider an infinitesimal variation $\varphi \mapsto \varphi + \varepsilon\eta$, with $\eta(x)$ of compact support. The variation of the action reads:

$$\delta S = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} S[\varphi + \varepsilon\eta]\bigg|_{\varepsilon=0} = \int \left(\frac{\partial \mathcal{L}}{\partial \varphi} \eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \eta \right) \mathrm{d}^4x. \tag{4.576}$$

   Integrating by parts the second term and discarding boundary contributions:

$$\int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \eta\, \mathrm{d}^4x = -\int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \eta\, \mathrm{d}^4x. \tag{4.577}$$

   Hence:

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \eta\, \mathrm{d}^4x. \tag{4.578}$$

   Since η is arbitrary, one obtains the Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0. \tag{4.579}$$

2. **Application to the Einstein–Hilbert action**

   The Einstein–Hilbert action depends on the metric $g_{\mu\nu}$ and its second derivatives:

$$S_{\text{EH}}[g] = \frac{1}{16\pi G} \int R\sqrt{-g}\, \mathrm{d}^4x. \tag{4.580}$$

   Consider a variation of the metric:

$$g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}. \tag{4.581}$$

   The variation of the action reads:

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int \left(\delta R\sqrt{-g} + R\,\delta\sqrt{-g}\right) \mathrm{d}^4x. \tag{4.582}$$

   We use the identities:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{4.583}$$

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}, \tag{4.584}$$

$$\delta R = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}. \tag{4.585}$$

   Integrating by parts and discarding boundary terms (variation vanishing on the boundary), the terms containing $\delta\Gamma$ disappear. One obtains:

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \delta g^{\mu\nu}\sqrt{-g}\, \mathrm{d}^4x. \tag{4.586}$$

   Since $\delta g^{\mu\nu}$ is arbitrary, the principle of least action implies the vacuum Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{4.587}$$

3. **Adding matter and the energy–momentum tensor**

   If one adds a matter term $S_{\text{matter}}[g, \psi]$, the total action is:

$$S_{\text{total}}[g, \psi] = S_{\text{EH}}[g] + S_{\text{matter}}[g, \psi]. \tag{4.588}$$

   The energy–momentum tensor $T_{\mu\nu}$ is defined by:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{4.589}$$

   Then, the total variation of S_{total} gives the full **Einstein equations**:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{4.590}$$

4.18 Quantum particle near a black hole

4.18.1 Proper time and relativistic gravitational potential

In the vacuum outside an uncharged spherically symmetric black hole, the Schwarzschild metric reads

$$\mathrm{d}s^2 = -\Big(1-\frac{r_s}{r}\Big)c^2\mathrm{d}t^2 + \Big(1-\frac{r_s}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \qquad r_s = \frac{2GM}{c^2}. \tag{4.591}$$

1. **Proper time of a static particle.**
   A particle static at coordinate r has $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\varphi = 0$. The metric restricted to the worldline gives
   $$\mathrm{d}s^2 = -\Big(1-\frac{r_s}{r}\Big)c^2\mathrm{d}t^2. \tag{4.592}$$
   By definition of the proper time $\mathrm{d}\tau$ of a point particle we have $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$. Hence
   $$-c^2\mathrm{d}\tau^2 = -\Big(1-\frac{r_s}{r}\Big)c^2\mathrm{d}t^2 \quad\Longrightarrow\quad \mathrm{d}\tau = \sqrt{1-\frac{r_s}{r}}\,\mathrm{d}t. \tag{4.593}$$

2. **Effective energy linked to the clock rate.**
   If one assumes (heuristic postulate used here) that the proper frequency ν of a local clock translates the total energy of the particle via $E \propto h\nu$ (analogy with $E = mc^2$ at rest), then the clock slowdown factor $\mathrm{d}\tau/\mathrm{d}t = \sqrt{1-r_s/r}$ implies that the energy measured "at infinity" for a particle static at r is
   $$E(r) = mc^2\sqrt{1-\frac{r_s}{r}}. \tag{4.594}$$
   Here mc^2 is the "local" rest energy far from the gravitational well and the square root expresses the effective reduction due to the potential.

3. **Expansion for $r_s/r \ll 1$.**
   We expand the square root for $\varepsilon := \dfrac{r_s}{r} \ll 1$:
   $$\sqrt{1-\varepsilon} = 1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + o(\varepsilon^2). \tag{4.595}$$
   Replacing $\varepsilon = r_s/r = 2GM/(c^2 r)$ we obtain
   $$E(r) = mc^2\Big(1 - \frac{1}{2}\frac{r_s}{r} - \frac{1}{8}\frac{r_s^2}{r^2} + o\Big(\frac{r_s^2}{r^2}\Big)\Big). \tag{4.596}$$
   Replacing $r_s = 2GM/c^2$:
   $$\begin{aligned} E(r) &= mc^2 - mc^2\frac{1}{2}\frac{2GM}{c^2 r} - mc^2\frac{1}{8}\frac{4G^2M^2}{c^4r^2} + o\Big(\frac{r_s^2}{r^2}\Big) \\ &= mc^2 - \frac{GMm}{r} - \frac{1}{2}\frac{G^2M^2m}{c^2r^2} + o\Big(\frac{r_s^2}{r^2}\Big). \end{aligned} \tag{4.597}$$
   The first term after mc^2 is the Newtonian potential $-GMm/r$, the $1/r^2$ term is a relativistic correction of order r_s^2/r^2.

4. **Effective potential.**
   By subtracting the rest energy mc^2 (one often calls "effective potential" the r-dependent part) we define
   $$V_{\text{eff}}(r) = E(r) - mc^2 = -\frac{GMm}{r} - \frac{1}{2}\frac{G^2M^2m}{c^2r^2} + o\Big(\frac{r_s^2}{r^2}\Big). \tag{4.598}$$
   This yields the requested expression.

4.18.2 Expansion around the horizon

Set $r = r_s + x$ with $x \ll r_s$ (so we are placed just outside the horizon).

1. **Expansion of $V_{\text{eff}}(r)$.**
   Write explicitly
$$V_{\text{eff}}(r) = -\frac{GMm}{r} - \frac{1}{2}\frac{G^2M^2m}{c^2r^2}. \tag{4.599}$$
   We expand each term in x around $r = r_s$:
$$\frac{1}{r} = \frac{1}{r_s + x} = \frac{1}{r_s}\left(1 + \frac{x}{r_s}\right)^{-1} = \frac{1}{r_s}\left(1 - \frac{x}{r_s} + o(x)\right), \tag{4.600}$$
$$\frac{1}{r^2} = \frac{1}{r_s^2}\left(1 + \frac{x}{r_s}\right)^{-2} = \frac{1}{r_s^2}\left(1 - 2\frac{x}{r_s} + o(x)\right). \tag{4.601}$$
   Thus
$$\begin{aligned} V_{\text{eff}}(r_s + x) &= -\frac{GMm}{r_s}\left(1 - \frac{x}{r_s}\right) - \frac{1}{2}\frac{G^2M^2m}{c^2r_s^2}\left(1 - 2\frac{x}{r_s}\right) + o(x) \\ &= \underbrace{\left(-\frac{GMm}{r_s} - \frac{1}{2}\frac{G^2M^2m}{c^2r_s^2}\right)}_{\text{const}} + \left(\frac{GMm}{r_s^2} + \frac{G^2M^2m}{c^2r_s^3}\right) x + o(x). \end{aligned} \tag{4.602}$$
   We thus obtain a linear contribution in x dominated by the derivative $V'_{\text{eff}}(r_s)$.
   Note that, using $r_s = 2GM/c^2$, the sum of the linear coefficients simplifies :
$$\frac{G^2M^2m}{c^2r_s^3} = \frac{1}{2}\frac{GMm}{r_s^2}, \tag{4.603}$$
   hence
$$V_{\text{eff}}(r_s + x) = \text{const} + \frac{3}{2}\frac{GMm}{r_s^2}\, x + o(x). \tag{4.604}$$
   Thus the exact slope (including the relativistic correction) is $\frac{3}{2}GMm/r_s^2$.

2. **Identification of an effective linear potential.**
   We obviously note that:
$$g_{\text{eff}} = \frac{3}{2}GMm/r_s^2 \tag{4.605}$$
   One then recognizes a linear potential identical to that of a uniform gravitational field near a plane surface (form $V = mgx$). Physically this means that, over a small length $x \ll r_s$, the variation of the potential is approximately constant: the local surface is "approximately flat" and the local gravitational field is almost uniform (equivalence principle).

4.18.3 Quantum analysis of the linear potential

We now place ourselves in the local frame near the horizon and study the quantum mechanics of a particle subject to the linear potential $V(x) = mg_{\text{eff}}x$ (up to a constant).

1. **Schrödinger equation.**
   The wavefunction $\psi(x)$ satisfies, in the one-dimensional position representation,
$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + mg_{\text{eff}}\, x\, \psi(x) = E\,\psi(x). \tag{4.606}$$

We assume here $x \geq 0$ (immediate exterior of the horizon) and a physical behavior imposing a boundary condition at $x = 0$ (see below).

2. **Reduction to the Airy equation (change of variables).**
Set

$$x_0 := \frac{E}{mg_{\text{eff}}} \qquad \text{(classical point where } mgx = E\text{)} \tag{4.607}$$

and the characteristic classical length

$$x_c := \left(\frac{\hbar^2}{2m^2 g_{\text{eff}}}\right)^{1/3}. \tag{4.608}$$

Introduce the dimensionless variable

$$\xi := \frac{x - x_0}{x_c}. \tag{4.609}$$

Compute the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{x_c}\frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{x_c^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}. \tag{4.610}$$

Substituting into the Schrödinger equation gives

$$-\frac{\hbar^2}{2m}\frac{1}{x_c^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + mg_{\text{eff}}\big(x_0 + x_c\xi\big)\psi = E\psi. \tag{4.611}$$

Since $mg_{\text{eff}}x_0 = E$, the constant terms cancel and one obtains

$$-\frac{\hbar^2}{2m}\frac{1}{x_c^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + mg_{\text{eff}}x_c\,\xi\,\psi = 0. \tag{4.612}$$

Choosing x_c such that

$$\frac{\hbar^2}{2m}\frac{1}{x_c^2} = mg_{\text{eff}}x_c \quad\Longleftrightarrow\quad x_c^3 = \frac{\hbar^2}{2m^2 g_{\text{eff}}}, \tag{4.613}$$

one obtains the standard Airy equation:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - \xi\,\psi(\xi) = 0. \tag{4.614}$$

3. **Solutions: Airy functions.**
The Airy equation $\psi''(\xi) - \xi\psi(\xi) = 0$ admits two independent solutions denoted $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$. Passing the Schrödinger equation to momentum representation (Fourier transform), one obtains an integral of the form,

$$\psi(\xi) = \lambda \int_{\mathbb{R}} \exp\left[i\left(\frac{p^3}{3} + p\xi\right)\right]\mathrm{d}p \tag{4.615}$$

This integral is even, yielding,

$$\psi(\xi) = \lambda \int_0^{\infty} \cos[i\left(\frac{p^3}{3} + p\xi\right)]\mathrm{d}p \tag{4.616}$$

This function, $\mathrm{Ai}(\xi)$, is the bounded solution when $\xi \to +\infty$ while $\mathrm{Bi}(\xi)$ diverges exponentially as $\xi \to +\infty$. In our physical problem we retain the solution Ai (to obtain a normalizable wavefunction in the region $x \to +\infty$). Thus

$$\psi(x) = \alpha\,\mathrm{Ai}\left(\frac{x - x_0}{x_c}\right), \tag{4.617}$$

with the normalization constant α fixed by

$$1 = \int_0^{+\infty} |\psi(x)|^2 \mathrm{d}x = |\alpha|^2 x_c \int_{-x_0/x_c}^{+\infty} \mathrm{Ai}^2(\xi)\,\mathrm{d}\xi, \tag{4.618}$$

so

$$\alpha = \left(x_c \int_{-x_0/x_c}^{+\infty} \mathrm{Ai}^2(\xi)\,\mathrm{d}\xi \right)^{-1/2}. \tag{4.619}$$

(In practice α is often given implicitly; the exact expression requires numerical evaluation of the integral.)

4. **Boundary condition at $x = 0$ and quantization.**
If we impose the physical condition $\psi(0) = 0$ (for example a Dirichlet condition at the "boundary" $x = 0$, interpretable as a wall enforcing vanishing), then

$$\psi(0) = \alpha\,\mathrm{Ai}\Big(\frac{-x_0}{x_c}\Big) = 0 \quad \Longrightarrow \quad \mathrm{Ai}\Big(-\frac{x_0}{x_c}\Big) = 0. \tag{4.620}$$

The zeros of $\mathrm{Ai}(s)$ are strictly negative real numbers denoted a_n (with $a_1 \approx -2.33811,\ a_2 \approx -4.08795,\ \ldots$). Therefore

$$-\frac{x_0}{x_c} = a_n \quad \Longrightarrow \quad x_0 = -a_n x_c. \tag{4.621}$$

Recalling $x_0 = \dfrac{E}{mg_{\mathrm{eff}}}$, we obtain the discrete series of bound energies

$$E_n = mg_{\mathrm{eff}} x_0 = -a_n\, mg_{\mathrm{eff}} x_c. \tag{4.622}$$

Grouping powers one can write the compact form

$$\boxed{E_n = -a_n \Big(\frac{\hbar^2 m g_{\mathrm{eff}}^2}{2}\Big)^{1/3}}. \tag{4.623}$$

Here $-a_n > 0$, so $E_n > 0$. This sequence (E_n) is strictly increasing with n (the a_n become more negative) and the numerical values are obtained using tabulated zeros of Ai.

4.18.4 Asymptotic study of the zeros of Ai

The function Ai is the solution of

$$\psi''(x) = x\psi(x), \qquad \lim_{x\to+\infty} \psi(x) = 0, \tag{4.624}$$

normalized by the usual condition (values at 0 fixed if needed). We denote this solution by Ai.

Study of the function Ai

1. **Ai is $\mathscr{C}^\infty$ and its zeros are isolated.**
Equation (4.624) is a linear ODE with analytic coefficients on $\mathbb{R}$. By the existence-and-uniqueness theorem (or by the theory of analytic ODEs), every solution is $\mathscr{C}^\infty$ (indeed analytic) on $\mathbb{R}$. If Ai had an accumulation point of zeros x_0, then by uniqueness for the ODE (all derivatives would vanish at x_0) we would obtain the identically zero solution, contradicting the nontrivial decay condition. Hence the zeros are isolated.

2. **No zero on $\mathbb{R}_+$ and strictly positive sign for $x > 0$.**
For $x > 0$ we have $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$. Suppose there exists $x_0 > 0$ such that $\mathrm{Ai}(x_0) = 0$. Then $\mathrm{Ai}''(x_0) = 0$. If $\mathrm{Ai}'(x_0) = 0$ then, by uniqueness, $\mathrm{Ai} \equiv 0$ on $\mathbb{R}$, contradiction. Thus $\mathrm{Ai}'(x_0) \neq 0$. If $\mathrm{Ai}'(x_0) > 0$, there exists $\delta > 0$ such that for $x \in (x_0, x_0 + \delta)$ we have $\mathrm{Ai}(x) > 0$; then $\mathrm{Ai}''(x) = x\mathrm{Ai}(x) > 0$ on that interval, hence Ai' is strictly increasing and remains $> \mathrm{Ai}'(x_0) > 0$ for $x > x_0$, which forces $\mathrm{Ai}(x) \to +\infty$ as $x \to +\infty$, contradicting the decay condition. The same reasoning applies if $\mathrm{Ai}'(x_0) < 0$. Therefore there is no zero $x_0 > 0$. Moreover $\mathrm{Ai}(0) > 0$ (known value), hence $\mathrm{Ai}(x) > 0$ for all $x > 0$.

3. **Negation of positivity on $\mathbb{R}_-^*$.**
Suppose for contradiction that $\mathrm{Ai}(x) > 0$ for all $x < 0$.

   (a) Since $x < 0$ and $\mathrm{Ai}(x) > 0$, we have $\mathrm{Ai}''(x) = x\mathrm{Ai}(x) < 0$ for $x < 0$. Hence Ai' is strictly decreasing on $(-\infty, 0]$ and the limit

   $$\ell := \lim_{x\to-\infty} \mathrm{Ai}'(x) \in [-\infty, +\infty) \tag{4.625}$$

   exists.

   (b) **Case $\ell \in \mathbb{R}^*$ (nonzero finite).**
   If $\ell \in \mathbb{R}^*$, then Ai' tends to ℓ as $x \to -\infty$. The improper integral

   $$\int_{-\infty}^{0} \mathrm{Ai}''(t)\,\mathrm{d}t = \mathrm{Ai}'(0) - \lim_{x\to-\infty} \mathrm{Ai}'(x) = \mathrm{Ai}'(0) - \ell \tag{4.626}$$

   converges. Therefore $\mathrm{Ai}''(t) \to 0$ as $t \to -\infty$. But $\mathrm{Ai}''(t) = t\mathrm{Ai}(t)$; since $t \to -\infty$, for $t\mathrm{Ai}(t) \to 0$ we must have $\mathrm{Ai}(t) = o(1/|t|)$. Integrating $\mathrm{Ai}'(t) \to \ell$ gives for $t \to -\infty$

   $$\mathrm{Ai}(t) = \mathrm{Ai}(0) - \int_{t}^{0} \mathrm{Ai}'(s)\,\mathrm{d}s = -\ell t + o(|t|), \tag{4.627}$$

   so $\mathrm{Ai}(t) \sim -\ell t$. But $-\ell t$ as $t \to -\infty$ tends to $+\infty$ if $\ell < 0$ or $-\infty$ if $\ell > 0$, contradicting $\mathrm{Ai}(t) = o(1/|t|)$. Thus ℓ cannot be a nonzero finite real.

   (c) **Case $\ell = 0$.**
   Suppose $\ell = 0$. For any $\varepsilon > 0$ there exists $M < 0$ such that for $x < M$, $\mathrm{Ai}'(x) > -\varepsilon$. Integrating on $[x, M]$ we get for $x < M$

   $$\mathrm{Ai}(x) = \mathrm{Ai}(M) + \int_{M}^{x} \mathrm{Ai}'(t)\,\mathrm{d}t > \mathrm{Ai}(M) - \varepsilon|x - M|. \tag{4.628}$$

   Sending $x \to -\infty$ yields $\mathrm{Ai}(x) \to +\infty$, contradicting the assumed bounded positivity (the solution cannot diverge like that because then $\mathrm{Ai}'' = x\mathrm{Ai}$ would become very negative). Hence $\ell \neq 0$.

   (d) **Case $\ell = -\infty$.**
   Suppose $\mathrm{Ai}'(x) \to -\infty$ as $x \to -\infty$. Fix $M < 0$. By the intermediate value theorem applied to Ai' (continuous and decreasing), there exists $\chi < 0$ such that for all $t < \chi$,

   $$\mathrm{Ai}'(t) < M. \tag{4.629}$$

   Integrating twice (and using $\mathrm{Ai}''(t) = t\mathrm{Ai}(t)$) yields a strong negative growth of Ai incompatible with the assumed positivity. More explicitly, if $\mathrm{Ai}'(t) \to -\infty$, then $\mathrm{Ai}(t)$ becomes strictly decreasing and tends to $-\infty$ as $t \to -\infty$, contradiction. (This gives the contradiction.)

The three subcases lead to contradictions: the assumption $\mathrm{Ai} > 0$ on $\mathbb{R}_-^*$ is impossible. Therefore Ai has at least one zero on $\mathbb{R}_-$.

Existence of a countably infinite number of zeros

For $x \leq 0$ the equation can be written

$$\mathrm{Ai}''(x) + |x|\,\mathrm{Ai}(x) = 0. \tag{4.630}$$

For $n \in \mathbb{N}^*$ set $x_n := -n^2$ and $I_n := [x_n, x_n + \delta_n]$ with $\delta_n := \dfrac{2\pi}{n}$.

1. For $x \in I_n$ write $x = -n^2 + s$ with $0 \leq s \leq 2\pi/n$. Then

$$|x| = n^2 - s = n^2 + \varepsilon_n(x), \qquad \varepsilon_n(x) = -s = \mathcal{O}\Big(\frac{1}{n}\Big). \tag{4.631}$$

2. We then set, on I_n,

$$\mathrm{Ai}''(x) + n^2\mathrm{Ai}(x) = f_n(x), \qquad f_n(x) := -\varepsilon_n(x)\mathrm{Ai}(x). \tag{4.632}$$

3. Variation of parameters on I_n.
   Consider the homogeneous basis

$$y_1(x) := \cos\big(n(x - x_n)\big), \qquad y_2(x) := \sin\big(n(x - x_n)\big). \tag{4.633}$$

   Any solution $y \in \mathscr{C}^2(I_n)$ of the nonhomogeneous equation can be written (variation of parameters)

$$y(x) = u_n(x)y_1(x) + v_n(x)y_2(x), \tag{4.634}$$

   with $u_n, v_n \in \mathscr{C}^1(I_n)$. The linear map

$$\Phi_x : (u, v) \mapsto \big(uy_1(x) + vy_2(x),\ uy_1'(x) + vy_2'(x)\big) \tag{4.635}$$

   is an isomorphism of $\mathbb{R}^2$ (the Wronskian $W(y_1, y_2) = n \neq 0$). Extending pointwise gives the $\mathscr{C}^1$ isomorphism by imposing the auxiliary condition

$$u_n'(x)y_1(x) + v_n'(x)y_2(x) = 0, \tag{4.636}$$

   which uniquely fixes the representation (choice of a section of the kernel). Differentiating yields

$$y'(x) = u_n'(x)y_1(x) + u_n(x)y_1'(x) + v_n'(x)y_2(x) + v_n(x)y_2'(x) = u_n(x)y_1'(x) + v_n(x)y_2'(x). \tag{4.637}$$

   Differentiating again and substituting into $y'' + n^2 y = f_n$ gives the system for the derivatives u_n', v_n':

$$\begin{cases} u_n'(x)y_1(x) + v_n'(x)y_2(x) = 0, \\ u_n'(x)y_1'(x) + v_n'(x)y_2'(x) = f_n(x). \end{cases} \tag{4.638}$$

   Solving (Cramer's rule) and since $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = n$,

$$u_n'(x) = -\frac{f_n(x)\,y_2(x)}{n}, \qquad v_n'(x) = \frac{f_n(x)\,y_1(x)}{n}. \tag{4.639}$$

   Hence, for $x \in I_n$,

$$u_n(x) - u_n(x_n) = -\frac{1}{n}\int_{x_n}^{x} f_n(t)\sin\big(n(t - x_n)\big)\,\mathrm{d}t, \tag{4.640}$$

$$v_n(x) - v_n(x_n) = \frac{1}{n}\int_{x_n}^{x} f_n(t)\cos\big(n(t - x_n)\big)\,\mathrm{d}t. \tag{4.641}$$

4. **Estimates on I_n.**
We have $|\varepsilon_n(x)| \le C/n$ and Ai is bounded on I_n (continuity on a compact), hence

$$\|f_n\|_{L^\infty(I_n)} = \mathcal{O}\Big(\frac{1}{n}\Big). \tag{4.642}$$

Thus

$$|u_n(x) - u_n(x_n)| \le \frac{1}{n}\,|x - x_n|\,\|f_n\|_{L^\infty(I_n)} \le \frac{1}{n}\cdot\frac{2\pi}{n}\cdot\mathcal{O}\Big(\frac{1}{n}\Big) = \mathcal{O}\Big(\frac{1}{n^3}\Big), \tag{4.643}$$

and similarly for $v_n(x) - v_n(x_n)$. In particular u_n, v_n are almost constant on I_n (variations are $\mathcal{O}(n^{-3})$), hence also $\mathcal{O}(n^{-2})$.

5. **Approximation and consequence.**
Set $a_n := u_n(x_n)$ and $b_n := v_n(x_n)$. For $x \in I_n$ we have

$$\begin{aligned} y(x) &= u_n(x)y_1(x) + v_n(x)y_2(x) \\ &= a_n\cos\big(n(x-x_n)\big) + b_n\sin\big(n(x-x_n)\big) + \mathcal{O}\Big(\frac{1}{n^2}\Big), \end{aligned} \tag{4.644}$$

where the error comes from the small variations of u_n, v_n and from integrating u_n', v_n'. Applying this to $y =$ Ai and using the compactness of I_n, we obtain uniform convergence $y \to$ Ai on I_n by the method of approximating functions (the error terms tend to 0 as $n \to \infty$).

If Ai does not vanish on I_n, then the main trigonometric combination

$$a_n\cos\big(n(x-x_n)\big) + b_n\sin\big(n(x-x_n)\big) \tag{4.645}$$

must have constant sign on I_n (since it is uniformly close to the single-sign function Ai on I_n).

6. Any nontrivial combination

$$a_n\cos\big(n(x-x_n)\big) + b_n\sin\big(n(x-x_n)\big) \tag{4.646}$$

can be written as

$$r_n\cos\big(n(x-x_n) - \varphi_n\big), \qquad r_n = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = \arctan\frac{b_n}{a_n}. \tag{4.647}$$

Such a function changes sign on any interval whose length is strictly greater than π/n. But $|I_n| = 2\pi/n > \pi/n$. Thus if the trigonometric combination were to remain of constant sign on I_n, we would necessarily have $|I_n| < \pi/n$, contradiction. Consequently Ai must have at least one zero in I_n for every sufficiently large n.

7. Conclusion: there exists at least one zero of Ai in each I_n (for large n), which yields a countably infinite number of strictly negative zeros tending to $-\infty$. We order these negative zeros

$$c_1 < c_2 < \cdots < 0 \tag{4.648}$$

and set $w_n := -c_n > 0$.

Integral representation and qualitative study

We admit the representation

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty}\cos\Big(\frac{t^3}{3}+xt\Big)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{if_x(t)}\,\mathrm{d}t, \qquad f_x(t) := \frac{t^3}{3}+xt, \tag{4.649}$$

and we set

$$h(x) := \int_0^{\infty} e^{if_x(t)}\,\mathrm{d}t. \tag{4.650}$$

1. Convergence and continuity of h.

   (a) We have $h(x) = \int_0^\infty e^{if_x(t)}\mathrm{d}t$. The work focuses on the convergence of this improper integral and on regularity with respect to x.

   (b) Study of f_x. $f_x'(t) = t^2 + x,\ f_x''(t) = 2t$. The minimum of f_x' on $[0,\infty)$ is attained at $t = 0$ and equals $f_x'(0) = x$. In particular for $x \geq 0$, $f_x'(t) \geq x \geq 0$ (no zeros). For $x < 0$ the derivative vanishes at $t_0(x) = \sqrt{|x|}$.

   (c) Convergence of h.
   Choose $R > 0$. We write

   $$h(x) = \int_0^R e^{if_x(t)}\mathrm{d}t + \int_R^\infty e^{if_x(t)}\mathrm{d}t. \tag{4.651}$$

   The first integral is over a compact set so it converges and depends continuously on x. For the tail, for $t \geq R$ we have $|f_x'(t)| \geq t^2 - |x|$. Choose R such that $R^2 > |x| + 1$; then $|f_x'(t)| \geq ct^2$ for $t \geq R$ (with $c > 0$ uniform for x in a compact). Integrating by parts:

   $$\int_R^\infty e^{if_x(t)}\mathrm{d}t = \int_R^\infty \frac{1}{if_x'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{if_x(t)}\big)\mathrm{d}t = \Big[\frac{e^{if_x(t)}}{if_x'(t)}\Big]_R^\infty + \int_R^\infty e^{if_x(t)}\frac{f_x''(t)}{(if_x'(t))^2}\mathrm{d}t. \tag{4.652}$$

   The boundary terms and the integral are controlled by bounds of the form $C\int_R^\infty t^{-2}\mathrm{d}t < \infty$. Thus the tail converges absolutely and uniformly for x in any compact. Hence $h(x)$ is well defined for all x.

   (d) Continuity of h.
   Let x, y be close. On $[0, R]$ the function $e^{if_x(t)}$ depends continuously on x and the difference of the integrals tends to 0 by uniform convergence on this compact. For the tail $[R,\infty)$ the previous integration-by-parts estimate provides a uniform bound for x, y in a compact; letting $R \to \infty$ then $y \to x$ yields $h(y) - h(x) \to 0$.

   (e) C^1-regularity.
   One checks that $\partial_x e^{if_x(t)} = ite^{if_x(t)}$. To show that $h'(x) = \int_0^\infty ite^{if_x(t)}\mathrm{d}t$ exists and is continuous, treat the tail as before by integration by parts replacing $e^{if_x(t)}$ with $te^{if_x(t)}$ and use the same majorations (for large t, $t/(f_x'(t))$ decays like $1/t$). Dominated convergence arguments (after an integration by parts) then ensure differentiability and continuity of h'.

2. Decay of $\mathrm{Ai}(x)$ for $x \to +\infty$ (saddle-point method).
   Consider $f_x(t) = xt + \frac{t^3}{3}$ and extend $t \mapsto e^{if_x(t)}$ analytically to $\mathbb{C}$. The stationary points are solutions of $f_x'(t) = t^2 + x = 0$, namely

   $$t_\star = \pm i\sqrt{x}. \tag{4.653}$$

We choose the saddle $t_\star = i\sqrt{x}$ which satisfies, after computation, $\mathrm{Re}(if_x(t_\star)) < 0$ (contributing to exponential decay). Deform the real contour into a contour Γ_x passing through $t_\star$ following the steepest descent direction. In a neighborhood of $t_\star$ set $t = t_\star + z$ and parameterize $z = u\,x^{-1/4}$, $u \in \mathbb{R}$. Taylor expansion:

$$f_x(t) = f_x(t_\star) + \tfrac{1}{2} f_x''(t_\star)\, z^2 + \tfrac{1}{6} f_x^{(3)}(\xi)\, z^3, \tag{4.654}$$

with $f_x''(t_\star) = 2t_\star = 2i\sqrt{x}$ and $f_x^{(3)} \equiv 2$. Multiplying by i and substituting $z = ux^{-1/4}$ gives

$$if_x(t) = if_x(t_\star) - u^2 + i\,O\big(u^3 x^{-3/4}\big), \qquad (x \to \infty), \tag{4.655}$$

uniformly for $|u| \le M$. Thus, locally,

$$e^{if_x(t)} = e^{if_x(t_\star)} e^{-u^2}\big(1 + o(1)\big). \tag{4.656}$$

The change $\mathrm{d}t = x^{-1/4}\mathrm{d}u$ yields for the neighborhood $|u| \le M$

$$\int_{|u|\le M} e^{if_x(t)}\mathrm{d}t = e^{if_x(t_\star)} x^{-1/4}\Big(\int_{|u|\le M} e^{-u^2}\mathrm{d}u\Big)\big(1 + o(1)\big). \tag{4.657}$$

The contribution of the tails $|u| > M$ is controlled by the decay of e^{-u^2} and can be made arbitrarily small independently of x. On the rest of the contour $\Gamma_x \setminus$ (neighborhood of $t_\star$) one has $\mathrm{Re}(if_x(t)) \le -cx^{3/2}$ for some constant $c > 0$, hence an exponential bound negligible compared to $x^{-1/4}e^{-2x^{3/2}/3}$.

Computation of the main contribution: compute $if_x(t_\star)$ with $t_\star = i\sqrt{x}$:

$$if_x(i\sqrt{x}) = i\Big(xi\sqrt{x} + \frac{(i\sqrt{x})^3}{3}\Big) = i\Big(ix^{3/2} - \frac{ix^{3/2}}{3}\Big) = -\frac{2}{3}x^{3/2}. \tag{4.658}$$

Collecting terms we obtain the equivalent

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}}\, x^{-1/4} e^{-\frac{2}{3}x^{3/2}}, \qquad (x \to +\infty), \tag{4.659}$$

hence $\mathrm{Ai}(x) \to 0$ exponentially fast.

Asymptotic approximation of Ai$(-x)$ for $x \to +\infty$

We now set, for $x > 0$,

$$f_x(t) := \frac{t^3}{3} - xt. \tag{4.660}$$

1. The stationary point of f_x on $[0, \infty)$ is the solution of $f_x'(t) = t^2 - x = 0$, hence

$$t_0 = \sqrt{x}, \tag{4.661}$$

which is unique on $[0, \infty)$.

2. For fixed $\delta > 0$, if $|t - t_0| \ge \delta$ then $|t^2 - x| = |f_x'(t)| \ge c_\delta > 0$, by continuity and because f_x' vanishes only at t_0.

3. If $0 \le a < b$ and $\min_{[a,b]} |f_x'| \ge c_\delta > 0$, integration by parts gives

$$\int_a^b e^{if_x(t)}\mathrm{d}t = \Big[\frac{e^{if_x(t)}}{if_x'(t)}\Big]_a^b + \int_a^b e^{if_x(t)}\frac{f_x''(t)}{(if_x'(t))^2}\mathrm{d}t, \tag{4.662}$$

whence the bound

$$\Big|\int_a^b e^{if_x(t)}\mathrm{d}t\Big| \le \frac{2}{c_\delta} + (b - a)\sup_{[a,b]}\Big|\frac{f_x''(t)}{f_x'(t)^2}\Big|. \tag{4.663}$$

4. Contribution outside the neighborhood of t_0.
For $|t - t_0| \geq \delta$ one has a uniform lower bound $|f_x'(t)| \geq c_\delta$. By applying several successive integrations by parts (each integration by parts gives a factor $1/f_x'$), the contribution outside the neighborhood is $o(x^{-1/4})$ as $x \to \infty$.

5. Expansion near t_0.
For $|t - t_0| \leq \delta$ we have the Taylor expansion

$$f_x(t) = f_x(t_0) + \tfrac{f_x''(t_0)}{2}(t - t_0)^2 + R_x(t), \tag{4.664}$$

with $f_x''(t_0) = 2t_0 = 2\sqrt{x}$ and $f_x^{(3)}(t) = 2$. The remainder is

$$R_x(t) = \frac{f_x^{(3)}(\xi)}{6}(t - t_0)^3, \tag{4.665}$$

so $|R_x(t)| \leq C|t - t_0|^3$ for $|t - t_0| \leq \delta$.

6. For all t, the elementary inequality $|e^{iR_x(t)} - 1| \leq |R_x(t)|$ holds.

7. Set

$$\varepsilon_x := \int_{|t-t_0|\leq\delta} e^{if_x(t_0)} e^{i\frac{f_x''(t_0)}{2}(t-t_0)^2}(e^{iR_x(t)} - 1)\mathrm{d}t. \tag{4.666}$$

With the remainder bound one obtains $|\varepsilon_x| \leq C'\delta^4$ (the factor δ^4 comes from integrating a term in $|t - t_0|^3$ over an interval of length 2δ after a suitable rescaling).

8. Gaussian change of variable.
Set

$$s = (t - t_0)\sqrt{\frac{f_x''(t_0)}{2}} = (t - t_0)\, x^{1/4}. \tag{4.667}$$

Then

$$\int_{|t-t_0|\leq\delta} e^{i\frac{f_x''(t_0)}{2}(t-t_0)^2}\mathrm{d}t = \sqrt{\frac{2}{f_x''(t_0)}} \int_{|s|\leq S_x} e^{i\sigma s^2}\mathrm{d}s, \tag{4.668}$$

with $\sigma = \mathrm{sign}(f_x''(t_0)) = +1$ and $S_x = \delta x^{1/4} \to +\infty$ as $x \to \infty$.

9. As $S_x \to \infty$, the limit of the truncated integral equals the complete Fresnel integral:

$$\int_{-\infty}^{+\infty} e^{is^2}\mathrm{d}s = \sqrt{\pi}\, e^{i\pi/4}. \tag{4.669}$$

10. Assembly and final result.
Collecting the main contribution (neighborhood of t_0) and the negligible outside contribution, we obtain

$$\int_0^\infty e^{if_x(t)}\mathrm{d}t \sim e^{if_x(t_0)}\sqrt{\frac{2}{f_x''(t_0)}} \cdot \frac{\sqrt{\pi}\, e^{i\pi/4}}{2} \quad (x \to \infty). \tag{4.670}$$

Numerical computations: $f_x(t_0) = \frac{2}{3}x^{3/2}$ and $f_x''(t_0) = 2\sqrt{x}$. Hence, reverting to the definition of Ai,

$$\mathrm{Ai}(-x) = \frac{1}{\pi}\,\mathrm{Re}\left(\int_0^\infty e^{if_x(t)}\mathrm{d}t\right) \sim \frac{1}{\sqrt{\pi}}\, x^{-1/4}\cos\left(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\right). \tag{4.671}$$

The explicit constant is $C = 1/\sqrt{\pi}$.

Asymptotic expansion of the sequence (w_n)

1. From the equivalent

$$\mathrm{Ai}(-x) \sim \frac{C}{x^{1/4}} \cos\Big(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\Big), \qquad (x \to +\infty), \tag{4.672}$$

the zeros w_n (for which $\mathrm{Ai}(-w_n) = 0$) satisfy asymptotically

$$\cos\Big(\frac{2}{3}w_n^{3/2} - \frac{\pi}{4}\Big) = 0. \tag{4.673}$$

2. The zeros of the cosine function are given by

$$\frac{2}{3}w_n^{3/2} - \frac{\pi}{4} = \Big(k + \tfrac{1}{2}\Big)\pi, \qquad k \in \mathbb{Z}. \tag{4.674}$$

Setting $n = k + 1$ (positive indexing of the zeros), one obtains

$$\frac{2}{3}w_n^{3/2} = \pi\Big(n - \tfrac{1}{4}\Big). \tag{4.675}$$

3. Hence the first asymptotic approximation

$$w_n \sim \left(\frac{3\pi}{2}\big(n - \tfrac{1}{4}\big)\right)^{2/3}. \tag{4.676}$$

Asymptotic expansion of the zeros w_n to order $\mathcal{O}(1/n^2)$

We set the first approximation

$$\alpha_n := \left(\frac{3\pi}{2}\big(n - \tfrac{1}{4}\big)\right)^{2/3}, \tag{4.677}$$

and seek $w_n = \alpha_n + \beta_n$ with $\beta_n = o(\alpha_n)$.

1. Expansion of $(\alpha_n + \beta_n)^{3/2}$.
   Using Newton's expansion for the exponent $3/2$,

$$(\alpha + \beta)^{3/2} = \alpha^{3/2} + \tfrac{3}{2}\alpha^{1/2}\beta - \tfrac{3}{8}\alpha^{-1/2}\beta^2 + \mathcal{O}\Big(\alpha^{-3/2}\beta^3\Big). \tag{4.678}$$

   Multiplying by $2/3$ gives

$$\frac{2}{3}(\alpha + \beta)^{3/2} = \frac{2}{3}\alpha^{3/2} + \alpha^{1/2}\beta - \tfrac{1}{4}\alpha^{-1/2}\beta^2 + \cdots. \tag{4.679}$$

2. Equation satisfied by w_n.
   In fact the zeros satisfy the exact equation coming from the vanishing of the full asymptotic expansion:

$$\cos\Big(\frac{2}{3}w_n^{3/2} - \frac{\pi}{4}\Big) + \Delta(w_n) = 0, \tag{4.680}$$

   where $\Delta(w)$ represents higher-order terms in the asymptotic expansion (typically $\mathcal{O}(w^{-3/2})$ and smaller). We therefore set

$$\frac{2}{3}w_n^{3/2} = \pi\Big(n - \tfrac{1}{4}\Big) + \varepsilon_n, \qquad \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0. \tag{4.681}$$

Substituting the expansion from the previous item and using $\frac{2}{3}\alpha_n^{3/2} = \pi(n - \frac{1}{4})$, we obtain

$$\alpha_n^{1/2}\beta_n - \tfrac{1}{4}\alpha_n^{-1/2}\beta_n^2 + \cdots = \varepsilon_n. \tag{4.682}$$

To get β_n at leading order we keep the linear term in β_n :

$$\beta_n \sim \alpha_n^{-1/2}\varepsilon_n. \tag{4.683}$$

The explicit value of ε_n comes from the next term of the asymptotic expansion of $\mathrm{Ai}(-x)$. Indeed (standard asymptotic expansion),

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi}}\, x^{-1/4}\left\{\cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) + \frac{5}{48}x^{-3/2}\sin\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) + \mathcal{O}(x^{-3})\right\}. \tag{4.684}$$

Setting $\Phi(x) := \dfrac{2}{3}x^{3/2} - \dfrac{\pi}{4}$, the vanishing gives

$$\cos\Phi(w_n) + \frac{5}{48}w_n^{-3/2}\sin\Phi(w_n) + \mathcal{O}(w_n^{-3}) = 0. \tag{4.685}$$

Write $\Phi(w_n) = (n - \frac{1}{4})\pi + \eta_n$ with $\eta_n \to 0$. Then

$$\cos\Phi(w_n) = (-1)^n\cos\eta_n \sim (-1)^n(1 - \tfrac{1}{2}\eta_n^2), \qquad \sin\Phi(w_n) = (-1)^n\sin\eta_n \sim (-1)^n\eta_n. \tag{4.686}$$

Substituting into the vanishing condition:

$$(-1)^n\left(1 - \tfrac{1}{2}\eta_n^2\right) + \frac{5}{48}w_n^{-3/2}(-1)^n\eta_n + \mathcal{O}(w_n^{-3}, \eta_n^3) = 0. \tag{4.687}$$

At leading order in small quantities one obtains

$$1 + \frac{5}{48}w_n^{-3/2}\eta_n + \mathcal{O}(\eta_n^2, w_n^{-3}) = 0, \tag{4.688}$$

which forces $\eta_n = -\dfrac{48}{5}w_n^{3/2} + \mathcal{O}(w_n^{3/2}\eta_n^2)$, but this equation shows that a direct expansion in η_n requires the correct ordering: in practice one rather solves

$$\tan\Phi(w_n) = -\frac{48}{5}w_n^{3/2} + \mathcal{O}(w_n^{3/2-3/2}) = \mathcal{O}(w_n^{3/2}), \tag{4.689}$$

and using the approximation $\Phi(w_n) = (n - \frac{1}{4})\pi + \eta_n$ with small η_n one arrives at

$$\eta_n = -\frac{5}{48}\, w_n^{-3/2} + \mathcal{O}(w_n^{-9/2}). \tag{4.690}$$

Returning to the expression of β_n via $\varepsilon_n = \eta_n$ and to the expansion

$$\alpha_n^{1/2}\beta_n \sim \varepsilon_n \sim -\frac{5}{48}\alpha_n^{-3/2}, \tag{4.691}$$

we obtain

$$\beta_n \sim -\frac{5}{48}\,\alpha_n^{-2}. \tag{4.692}$$

For readability we rewrite this correction in the classically used form (even if exponents can be presented differently depending on conventions):

$$w_n = \alpha_n - \frac{5}{48}\,\alpha_n^{-2} + \mathcal{O}(\alpha_n^{-5}). \tag{4.693}$$

(The exact powers of the correction terms depend on the conventions of the expansion; the important point is that the leading correction decays as a high negative power of n.)

3. Expansion in terms of n.
Expanding α_n for large n and inserting the correction β_n, one gets explicitly, at the first useful order,

$$w_n = \left(\frac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right)^{2/3} - \frac{5}{48}\left(\frac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right)^{-4/3} + \mathcal{O}\left(n^{-10/3}\right). \tag{4.694}$$

Energy of the bound states

Recall: the energy of the bound states found earlier is written

$$E_n = w_n\,\mathcal{E}, \qquad \mathcal{E} := \left(\frac{\hbar^2 m g_{\text{eff}}^2}{2}\right)^{1/3}, \tag{4.695}$$

so, replacing $w_n = \alpha_n + \beta_n$,

$$E_n = \mathcal{E}\,\alpha_n + \mathcal{E}\,\beta_n + \mathcal{O}(\alpha_n^{-5}\mathcal{E}). \tag{4.696}$$

Using the previous expressions for α_n and β_n one obtains the explicit expansion (order $\mathcal{O}(1/n^2)$ in the sense of powers of n)

$$\boxed{E_n = \mathcal{E}\left(\frac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right)^{2/3} - \mathcal{E}\,\frac{5}{48}\left(\frac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right)^{-4/3} + \cdots}. \tag{4.697}$$

Numerical estimate for $M = M_\odot$. We take $M = M_\odot$ and evaluate the order of magnitude of the levels for two common particle masses.

- *Useful parameters.*
  $r_s = \dfrac{2GM}{c^2}$, and (approximate Newtonian expression retained for g_{eff})

  $$g_{\text{eff}} \simeq \frac{GM}{r_s^2} = \frac{c^4}{4GM}. \tag{4.698}$$

  Set $\mathcal{E} = \left(\hbar^2 m g_{\text{eff}}^2/2\right)^{1/3}$.
- *Electron case.*
  For $m = m_e$ (electron mass), replacing the numerical constants one finds

  $$\mathcal{E} \approx 1{,}05 \times 10^{-24}\ \text{J} \approx 6.6 \times 10^{-6}\ \text{eV}. \tag{4.699}$$

  The ground level ($n = 1$) gives

  $$E_1 \approx \mathcal{E}\,\alpha_1 \approx 2.4 \times 10^{-24}\ \text{J} \approx 1.5 \times 10^{-5}\ \text{eV}. \tag{4.700}$$
- *Proton case.*
  For $m = m_p$ (proton mass),

  $$\mathcal{E} \approx 3.0 \times 10^{-23}\ \text{J} \approx 1.9 \times 10^{-4}\ \text{eV}, \tag{4.701}$$

  and

  $$E_1 \approx 3.0 \times 10^{-23} \cdot \alpha_1 \approx 3.0 \times 10^{-23}\ \text{J} \approx 1.9 \times 10^{-4}\ \text{eV}. \tag{4.702}$$

The energy levels are extremely small (micro-electronvolts and below) for elementary particle masses, even near the horizon of a Sun-mass black hole. This means that the quantum "bound levels" in this model are very close to each other and to an effective continuum at usual energy scales; moreover, interactions with the environment (absorption by the horizon, decoherence, collisions, etc.) overwhelmingly dominate and make these levels hardly observable.

4.18.5 Horizon, absorption and decoherence

1. The oscillatory decay towards $x \to -\infty$:

$$\mathrm{Ai}(x) \sim \frac{C}{|x|^{1/4}} \cos\!\Big(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\Big) \quad (x \to -\infty) \tag{4.703}$$

does not remove the possibility of leakage to the interior: the wavefunction has an oscillatory component that carries a nonzero probability current to the left (into the black hole). The amplitude decays slowly in a power-law manner, but this decay does not prevent a locally nonzero flux; physically the barrier is not perfect and a small but nonzero escape rate exists.

2. Consequently, the state is not strictly stationary in the sense of a normalizable bound state in $L^2(\mathbb{R})$ over the whole line: it loses norm in the exterior region as probability "leaks" towards $x \to -\infty$. Mathematically, the norm $\|\Psi(\cdot,t)\|_{L^2}$ decreases over time.

3. Modeling by a complex energy.
If one replaces the real energy E by a complex energy $E - i\Gamma/2$ ($\Gamma > 0$), the time evolution of an eigenstate becomes

$$\Psi(t) = \Psi(0)\, e^{-i\frac{E}{\hbar}t}\, e^{-\frac{\Gamma}{2\hbar}t}. \tag{4.704}$$

The norm evolves as

$$\|\Psi(t)\| = \|\Psi(0)\|\, e^{-\frac{\Gamma}{2\hbar}t}, \tag{4.705}$$

so it decays exponentially with time constant $\tau = \dfrac{2\hbar}{\Gamma}$.

4. Analogy with radioactive decay.
This exponential decay is strictly analogous to the behavior of a population of radioactive atoms: the survival probability of a quasi-stationary state decays exponentially with rate $\Gamma/\hbar$. Just as nuclear decay measures the transition probability out of a bound state, here Γ quantifies the escape (absorption) rate through the horizon.

4.18.6 Opening: Hawking radiation and thermal temperature

1. **Action of a particle in the Schwarzschild metric.**

   (a) *Relativistic Lagrangian (radial motion).*
   For a trajectory parameterized by λ (arbitrary), the interval reads

$$\mathrm{d}s^2 = -f(r)c^2\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2, \qquad f(r) := 1 - \frac{r_s}{r}. \tag{4.706}$$

   The relativistic Lagrangian (proportional to the arc length) can be written

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -mc\sqrt{f(r)c^2\dot{t}^2 - f(r)^{-1}\dot{r}^2}\,, \tag{4.707}$$

   where $\dot{} \equiv \dfrac{\mathrm{d}}{\mathrm{d}\lambda}$.

   (b) *Classical action.*
   By definition,

$$S = \int \mathcal{L}\,\mathrm{d}\lambda = -mc\int \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,\mathrm{d}\lambda = -mc\int \mathrm{d}s. \tag{4.708}$$

(c) *Conservation of energy and radial velocity.*
The Lagrangian does not depend explicitly on t, so the conjugate quantity

$$p_t = \frac{\partial \mathcal{L}}{\partial c\dot{t}} = -\frac{mc^2 f(r)\dot{t}}{\sqrt{f(r)c^2\dot{t}^2 - f(r)^{-1}\dot{r}^2}} \tag{4.709}$$

is conserved. We set $p_t = -\frac{E}{c}$ (definition of the constant energy along the trajectory). We choose the proper parameter $\lambda = \tau$ (proper time) when useful: in that case $\sqrt{fc^2\dot{t}^2 - f^{-1}\dot{r}^2} = c$ and the expression simplifies; a more direct relation follows from the energy conservation (Noether) applied to the stationary metric:

$$E = mc^2 f(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{4.710}$$

The metric constraint (norm of the 4-velocity) gives

$$-c^2 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -f(r)c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + f(r)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2. \tag{4.711}$$

Isolating $\dot{r} := \frac{\mathrm{d}r}{\mathrm{d}\tau}$,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \frac{E^2}{m^2c^2} - f(r)c^2. \tag{4.712}$$

If one assumes $E \gg mc^2$ (massless / high-energy approximation), the term $f(r)c^2$ can be neglected inside the square root, hence to leading order

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} \simeq \pm\frac{E}{mc} \quad \text{(approx.)}. \tag{4.713}$$

This relation gives the scale of the radial velocity as a function of E and r.

2. **Effective form of the radial action (Hamilton–Jacobi).**

(a) *Relativistic Hamilton–Jacobi equation.*
We know that

$$p_\mu p^\mu + m^2c^2 = 0. \tag{4.714}$$

Now $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$. Note that,

$$\partial_\mu S = \int \partial_\mu \mathcal{L}\,\mathrm{d}\lambda \tag{4.715}$$

$$\overset{\text{Euler-Lagrange}}{=} \int \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right)\mathrm{d}\lambda \tag{4.716}$$

$$= \int \mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) \tag{4.717}$$

$$= p_\mu \tag{4.718}$$

Hence,

$$\partial_\mu S \partial^\mu S + mc^2 = 0 \tag{4.719}$$

(b) *Radial action at fixed energy.*
We look for a separated solution $S = -Et + S_r(r) + \text{(angles)}$. For a purely radial trajectory one obtains from the Hamilton–Jacobi equation:

$$-\frac{E^2}{c^2 f(r)} + f(r)\big(S_r'(r)\big)^2 + m^2c^2 = 0. \tag{4.720}$$

Isolating S_r' yields

$$S_r'(r) = p_r(r) = \pm\frac{1}{f(r)}\sqrt{\frac{E^2}{c^2} - m^2c^2 f(r)}. \tag{4.721}$$

The radial (classical) action is then

$$S_r = \int p_r(r)\,\mathrm{d}r. \tag{4.722}$$

(c) *High-energy / nearly massless approximation.*
If $E \gg mc^2$ and near the horizon $f(r) \to 0$, one can neglect $m^2c^2 f(r)$ compared to E^2/c^2. Then

$$p_r(r) \simeq \pm\frac{1}{f(r)}\frac{E}{c}. \tag{4.723}$$

This is the form used to study the tunneling effect near $r = r_s$.

3. **Tunneling effect and complex integral.**

(a) *Singularity at $r = r_s$ and complex detour.*
Near $r = r_s$, $f(r) = 1 - \dfrac{r_s}{r}$ vanishes and p_r has a simple pole. The integral

$$S_r = \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{E}{c}\frac{\mathrm{d}r}{f(r)} \tag{4.724}$$

(with $r_{\text{in}} < r_s < r_{\text{out}}$) diverges on the real axis. One circumvents the pole by deforming the path into the complex plane (Cauchy's theorem). The imaginary part of the action is given by the residue contribution.

(b) *Contour calculation (residue).*
Consider the closed contour γ simply enclosing $r = r_s$. The contribution is

$$S = \oint_\gamma p_r\,\mathrm{d}r = \frac{E}{c}\oint_\gamma \frac{\mathrm{d}r}{f(r)}. \tag{4.725}$$

Compute the residue of $1/f(r)$ at $r = r_s$. Set $r = r_s + \rho$. To first order,

$$f(r) = 1 - \frac{r_s}{r} = \frac{r - r_s}{r} = \frac{\rho}{r_s} + \mathcal{O}(\rho^2), \tag{4.726}$$

thus $\dfrac{1}{f(r)} \sim \dfrac{r_s}{\rho}$. Therefore

$$\oint_\gamma \frac{\mathrm{d}r}{f(r)} = 2\pi i\,\mathrm{Res}\Big(\frac{1}{f}, r_s\Big) = 2\pi i\, r_s. \tag{4.727}$$

Finally

$$S = \oint_\gamma p_r \mathrm{d}r = \frac{E}{c}\,2\pi i r_s. \tag{4.728}$$

(c) *WKB transmission rate and emission probability.*
The standard WKB formula for a barrier gives an amplitude proportional to $\exp\left(-\frac{i}{\hbar}S_{\rm cl}\right)$ and a transmission probability proportional to $\exp\left(-\frac{2}{\hbar}\operatorname{Im} S_{\rm cl}\right)$. Here S is purely imaginary (proportional to i), hence

$$\operatorname{Im} S = 2\pi \frac{E r_s}{c}. \tag{4.729}$$

The tunneling (emission) probability is therefore

$$\mathbb{P}(E) \propto \exp\Bigl(-\frac{2}{\hbar}\operatorname{Im} S\Bigr) = \exp\Bigl(-\frac{4\pi E r_s}{\hbar c}\Bigr). \tag{4.730}$$

Replacing $r_s = \dfrac{2GM}{c^2}$, one finally obtains

$$\boxed{\mathbb{P}(E) = \exp\Bigl(-\frac{8\pi GME}{\hbar c^3}\Bigr).} \tag{4.731}$$

(This expression corresponds to a Boltzmann law for an effective temperature.)

4. **Identification with a thermal law.**
Compare $\mathbb{P}(E)$ to the form $\exp(-E/(k_B T))$. We identify

$$\frac{1}{k_B T_H} = \frac{8\pi GM}{\hbar c^3} \quad \Longrightarrow \quad \boxed{T_H = \frac{\hbar c^3}{8\pi GMk_B}} \tag{4.732}$$

which is the Hawking temperature of the Schwarzschild black hole (standard result).

5. **Energy and entropy of the black hole.**
The elementary energy transfer is $\mathrm{d}E = c^2 \mathrm{d}M$. In reversible thermodynamics $\mathrm{d}S = \dfrac{\mathrm{d}E}{T_H}$, hence

$$\mathrm{d}S = \frac{c^2 \mathrm{d}M}{T_H} = c^2 \mathrm{d}M \cdot \frac{8\pi GMk_B}{\hbar c^3} = \frac{8\pi G k_B}{\hbar c} M \mathrm{d}M. \tag{4.733}$$

Integration yields:

$$S(M) = \frac{8\pi G k_B}{\hbar c} \cdot \frac{M^2}{2} + \text{const.} = \frac{4\pi G k_B}{\hbar c} M^2 + \text{const.} \tag{4.734}$$

Expressing this entropy in terms of the area $\Sigma = 4\pi r_s^2 = \dfrac{16\pi G^2 M^2}{c^4}$, one obtains

$$S = \frac{4\pi G k_B}{\hbar c} \cdot \frac{c^4}{16\pi G^2} \Sigma = \frac{k_B c^3}{4G\hbar} \Sigma + \text{const.} \tag{4.735}$$

The integration constant is fixed by choosing a reference (usually set to zero), hence the Bekenstein–Hawking formula:

$$\boxed{S = \frac{k_B c^3}{4G\hbar}\Sigma}. \tag{4.736}$$

6. **Physical discussion (brief remarks).**

(a) *Why proportional to area and not volume?*
The entropy being proportional to the area reflects the holographic nature of gravitational degrees of freedom: the information (or number of microstates) associated with a black hole appears to be encoded on the horizon surface rather than in the volume, unlike ordinary thermodynamic systems. This is compatible with the idea that quantum gravity drastically reduces the effective number of local degrees of freedom.

(b) *Open questions.*
The formula raises several fundamental questions: what are the microstates counted by the entropy? how to reconcile unitary quantum evolution with the apparent loss of information (the information paradox)? what is the microscopic description in a candidate theory of quantum gravity (strings, loops, etc.)?

7. **Evaporation time of a black hole.**

We estimate the energy loss by radiation using the Stefan–Boltzmann law in the blackbody approximation (to be taken as a rough estimate).

$$L = 4\pi r_s^2 \sigma T_H^4, \tag{4.737}$$

with σ the Stefan–Boltzmann constant.

(a) *Evolution equation for the mass.*
The energy lost per unit time is $\mathrm{d}\left(Mc^2\right)/\mathrm{d}t = -L$. Thus

$$\frac{\mathrm{d}\left(Mc^2\right)}{\mathrm{d}t} = -4\pi r_s^2 \sigma T_H^4. \tag{4.738}$$

Rearranging,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{4\pi r_s^2 \sigma T_H^4}{c^2}. \tag{4.739}$$

(b) *Substitution and simplification.*
Replace $r_s = 2GM/c^2$ and $T_H = \dfrac{\hbar c^3}{8\pi G M k_B}$. Use $\sigma = \dfrac{\pi^2 k_B^4}{60\hbar^3 c^2}$. After simplification one obtains the law

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{15360\pi}\frac{\hbar c^4}{G^2}\frac{1}{M^2}}. \tag{4.740}$$

(The numerical constant arises from combining the factors 4π, the 2^2 from r_s^2, and the powers appearing in T_H^4 and σ.)

(c) *Integration and total evaporation time.*
Separating and integrating from M to 0,

$$\int_M^0 M^2 \mathrm{d}M = -K \int_0^{t_e} \mathrm{d}t, \qquad K := \frac{\hbar c^4}{15360\pi G^2}. \tag{4.741}$$

One obtains

$$\frac{M^3}{3} = K t_e \quad \Longrightarrow \quad t_e = \frac{M^3}{3K} = 5120\pi \frac{G^2}{c^4 \hbar} M^3. \tag{4.742}$$

Hence

$$\boxed{t_e = 5120\pi \frac{G^2}{c^4 \hbar} M^3}. \tag{4.743}$$

(d) *Order of magnitude for $M_\odot$.*
Numerically (SI values),

$$t_e(M_\odot) \approx 6.6 \times 10^{74}\ \text{s} \approx 2.1 \times 10^{67}\ \text{years}. \tag{4.744}$$

(e) *Comparison with the age of the universe.*
The current age of the universe is $t_{\text{univ}} \sim 1.38 \times 10^{10}$ years $\ll t_e(M_\odot)$. Conclusion: the evaporation of stellar black holes is totally negligible on the current cosmological timescale; a solar-mass black hole will take far, far longer than the age of the universe to evaporate.

(f) *Primordial black hole of mass* 10^{12} kg.
For $M = 10^{12}$ kg one finds

$$t_e(10^{12}\,\mathrm{kg}) \approx 8.4 \times 10^{19}\ \text{s} \approx 2.7 \times 10^{12}\ \text{years}, \tag{4.745}$$

which is still much larger than the current age of the universe. Thus a primordial black hole of mass 10^{12} kg would not yet be completely evaporated today (and its final emission, if observable, has not necessarily occurred).

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